

Hidden Symmetries of Large N QCD

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Abstract

The new symmetry of the loop dynamics of QCD is found. This is a local SUSY $\delta s = \theta\beta(s)$, $\delta\theta = \beta(s)$ of the superloop field $X_\mu(s, \theta) = x_\mu(s) + \theta\psi_\mu(s)$. The remarkable thing is, there is no einbein-gravitino on this theory, which makes it a 1D topological supergravity, or locally SUSY quantum mechanics. Using this symmetry, we derive the large N_c loop equation in momentum superloop space. Introducing as before the position operator \hat{X}_μ we argue that the superloop equation is equivalent to invariance of correlation functions of products of these operators with respect to certain quadrilinear transformation. As a consequence of this nonlinear symmetry, the coefficients of the Voiculesku expansion of the position operator satisfy recurrent equations. The generators of this symmetry are involved in the glueball spectrum, as it follows from the loop-loop correlation function equation. The correlation functions of external flavor currents in the background of constant flavor gauge field with finite density of topological charge in the chiral limit are also considered. We represent them as certain finite dimensional integrals in superspace, involving the Greens function, which is expressed in terms of \hat{X}_μ . The expansion in powers of external momenta $k_1 \dots k_n$ is *calculable* in terms of the above universal numbers in the Voiculesku expansion.

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1 Introduction

Recently [1], some progress was achieved in a long-standing problem of string representation of large N QCD. We introduced the position operator \hat{X}_μ of the endpoint of the QCD string and applied the momentum loop equation [8, 9] (MLE) to get some relations between the planar connected moments of vacuum expectation values of this operator.

The non-commutative probability theory [6, 5, 7] was then used to build the Fock space and operator expansion of \hat{X}_μ in powers of the creation operator a_μ^\dagger which obeys the Cuntz algebra $a_\mu a_\nu^\dagger = \delta_{\mu\nu}$. The Fock space consists of words $a_{\mu_1}^\dagger a_{\mu_2}^\dagger \dots a_{\mu_n}^\dagger |0\rangle$, where the vacuum $|0\rangle$ is annihilated by a_μ .

There are several problems with this approach. One problem is that some coefficients of this operator expansion

$$\hat{X}_\mu = a_\mu + Aa_\mu^\dagger + Ba_\nu^\dagger a_\mu^\dagger a_\nu^\dagger + C(a_\nu^\dagger a_\nu^\dagger a_\mu^\dagger + a_\mu^\dagger a_\nu^\dagger a_\nu^\dagger) + \dots \quad (1)$$

are left undetermined. Apparently, some information is still missing. Another problem is that there seems to be no direct relation between these coefficients and the QCD observables.

As for the first problem, it was known since the first papers on the loop equation [2, 3] that one should restrict oneself to the class of so-called Stokes type functionals to reconstruct the Wilson loop from the loop equation. In those papers it was shown that iterations of the loop equation within this class of functionals reproduce the Faddeev-Popov perturbation theory including ghost loops.

The Stokes type functionals by definition satisfy the Mandelstam relation

$$\frac{\delta}{\delta x_\mu(s)} = x'_\nu(s) \frac{\partial}{\partial \sigma_{\mu\nu}(s)}, \quad (2)$$

where the area derivative $\frac{\partial}{\partial \sigma_{\mu\nu}(s)}$ is a skew symmetric tensor operator satisfying the Bianchi identity

$$\partial_\alpha(s) \frac{\partial}{\partial \sigma_{\beta\gamma}(s)} + \text{cyclic} = 0. \quad (3)$$

The derivative $\partial_\mu(s)$ is defined as

$$\partial_\mu(s) = \int_{s-0}^{s+0} dt \frac{\delta}{\delta x_\mu(t)}. \quad (4)$$

These relations were not taken into account in our previous paper [1]. Also, the loop equation used in that paper was not the most general one. It corresponded to the original vector loop equation [2] multiplied by the tangent vector $x'_\nu(s)$ and integrated around the loop. Normal component terms were eliminated by this projection. For example, in two dimensions, the loop averages were computed [4] using the normal component of the loop equation. The tangent components in this case were identically satisfied for the area dependent Ansatz.

The second problem is more serious. The operator \hat{X} as defined is apparently not an observable, so it may be infinite or zero in the local limit of the theory.

In this paper we fix both of these problems at the same time. We choose the superloop formalism (see, e.g. [10, 11]) which allows one to impose Mandelstam and Bianchi relations as local supersymmetry relations. We derive the large N superloop equation in momentum space.

The SuperFourier transformation we use here, is free of the zero point fluctuation problems, which were a serious obstacle to the old momentum loop dynamics. These fluctuations cancel by themselves, without any Wiener measure, by virtue of supersymmetry.

Thus, there are no more hidden infinite renormalization factors in our momentum superloops. The corresponding supersymmetric generalization of the position operator of the string obeys a different set of recurrent equations, which are more restrictive than before.

These relations are based on the important observation that there is a local SUSY in the loop dynamics. The local SUSY is valid in the large N_c loop equations as well as in the relations for these observables in the chiral limit. We are able to explicitly find all the superspace-dependent quantities (they reduce to the SUSY Θ functions).

The physical interpretation of these relations is found. They represent the *symmetry* relations for an operator \hat{X} correlation functions. One can perform certain nonlinear transformation of \hat{X} operator with two vector parameters. This symmetry is *completely* equivalent to the momentum loop equation.

The non-commuting vector generators of this symmetry act as “QCD string” endpoint translations. Their eigenfunctions must have fixed spatial momentum, which condition yields the glueball spectrum equation. We derive this condition from the glueball superloop equation.

The superloop in momentum space has direct physical meaning in the chiral limit of QCD. It is related to the correlation functions of external flavor currents. The local supermomentum is expressed in terms of the set of external momenta entering the quark loop.

One set of correlation functions involves the chiral anomaly (the periodic boundary conditions for the Fermi component of the superpath). Those are expected to be completely dominated by the perturbative contribution. The argument is based on the index theorem. However, in presence of non-smooth fluctuations of the gluon field this perturbative argument may fail. We relate the coefficients in the low energy expansion of these correlation functions to the coefficients of the Voiculesku expansion.

Another set of calculable observables is given by the ordinary correlation functions of external flavor currents taken in the background of external field with finite density of topological charge (the anti-periodic boundary conditions for the Fermi component of the superpath). The coefficients of the low energy expansion are also related to the coefficients of the Voiculesku expansion. We derive an explicit set of operator equations in presence of external field.

The resulting theory can be reformulated as the theory of the quark superfield propagating around the superloop, rather than target space. The corresponding propagator can be found exactly in terms of the quark position operator \hat{X}_μ . This resembles the string theory, but we do not see any precise correspondence.

In order to handle the chiral symmetry breaking in absence of the external density of topological charge we included the quark mass term. This resulted in extra term in the quark effective action on a superloop. This term appears to be a quartic interaction, rather than a mass term. This superfield theory is essentially one dimensional, so it will hopefully present no new problems.

2 Vacuum Energy and Superloops

Let us present our superloop formulation of QCD. It starts with the formula (derived in Appendix A) for the quark loop energy in the presence of color and flavor fields

$$\mathcal{E} = \text{const} - \frac{1}{2} \text{tr} \log \left(m_0^2 + (\imath \gamma_\mu \nabla_\mu)^2 \right) = \frac{1}{2} \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) e^{-T m_0^2} \text{tr} \exp \left(-T (\imath \gamma_\mu \nabla_\mu)^2 \right). \quad (5)$$

In this and subsequent formulas the limit $\alpha \rightarrow +0$ is understood. This α prescription is equivalent to the ζ regularization and it goes along with dimensional regularization.

The covariant derivative operator involves all the gauge potentials $\nabla_\mu = \partial_\mu + A_\mu + B_\mu$, where $A_\mu(x)$ is the quantum gluon field, and $B_\mu(x)$ is the external flavor field.

The SUSY follows from the effective Hamiltonian

$$(\imath \gamma_\mu \nabla_\mu)^2 = \{Q_L, Q_R\}, \quad Q_{L,R} = (\imath \gamma_\mu \nabla_\mu) \frac{1 \pm \gamma_5}{2}. \quad (6)$$

Therefore a Dirac operator generates the (global) SUSY transformations. This is the well-known SUSY Quantum Mechanics.

The path integral representation can be formulated in terms of the super trajectory $x_\mu(s), \psi_\mu(s)$. The Fermi coordinates $\psi_\mu(s)$ represent the the matrices $\gamma_\mu \sqrt{T}$.

It is most convenient to use the superspace $S = (s, \theta)$ with the superfield

$$X_\mu(S) = x_\mu(s) + \theta \psi_\mu(s). \quad (7)$$

with boundary conditions of (anti)periodicity

$$x_\mu(1) = x_\mu(0), \psi_\mu(1) = -\psi_\mu(0). \quad (8)$$

The negative sign for ψ_μ is needed to make the traces of γ matrices cyclic symmetric in spite of anti-commutation of ψ_μ . Moving the left ψ_μ to the right all the way through remaining (odd) number of ψ would produce the minus sign, unless we compensate it by the extra sign flip at $s = 1$.

The action is given by an integral over superloop

$$\mathcal{I} = \int_0^1 ds \int d\theta \mathcal{L}(S) \quad (9)$$

with the super Lagrangian (in our normalization of proper time and ψ fields)

$$\begin{aligned}\mathcal{L}(S) &= \mathcal{L}^{kin}(S) \oplus DX_\mu(S)A_\mu(X(S)) \oplus DX_\mu(S)B_\mu(X(S)), \\ \mathcal{L}^{kin}(S) &= -\frac{1}{4T} DX_\mu(S)D^2X_\mu(S)\end{aligned}\tag{10}$$

Here

$$D = \frac{\partial}{\partial\theta} + \theta \frac{\partial}{\partial s},\tag{11}$$

is a covariant derivative in superspace. Its square is the time derivative

$$D^2 = \frac{\partial}{\partial s}.\tag{12}$$

The direct sum of the color and flavor matrices is understood in the sense of taking the ordered exponential of the action in the (super)path integral

$$\begin{aligned}\int \mathcal{D}X \hat{T} \exp(\mathcal{I}) &= \int \mathcal{D}X \exp\left(\int_0^1 ds \int d\theta \mathcal{L}^{kin}(S)\right) \\ \hat{T} \exp\left(\int_0^1 ds \int d\theta DX_\mu(S) (A_\mu(X(S)) \oplus B_\mu(X(S)))\right)\end{aligned}\tag{13}$$

which is the non-Abelian generalization of the Feynman's path integral. One may prove that the results are equivalent to the more traditional approach with the ordinary exponential of a second quantized action.

Conventional formalism shifts the momentum by $\imath p_\mu \Rightarrow \imath p_\mu \oplus A_\mu \oplus B_\mu$ after which the gauge fields go to kinetic terms in the Lagrangian, which is inconvenient. So we undid this shift, coming back to original ideas of Feynman, who started with the (Abelian) Wilson loop as an amplitude for electron propagation. In our previous paper [8] we discussed these issues in detail.

3 Superloop Kinematics

In our case it is implied that the gauge fields have the superfield as their argument, which means that first derivative terms are also included. The (global) SUSY generalization of the ordered product is also known [11]. One introduces a SUSY generalization of the distance

$$L(S, S') = s' - s + \theta\theta',\tag{14}$$

and the corresponding theta function

$$\Theta(S, S') = \Theta(L(S, S')) = \Theta(s' - s) + \theta\theta'\delta(s' - s).\tag{15}$$

The covariant derivative $D = \frac{\partial}{\partial\theta} + \theta \frac{\partial}{\partial s}$ of this Θ function reduces to the SUSY δ function

$$D\Theta(S, S') = (\theta' - \theta)\delta(s' - s) \equiv \delta(S' - S).\tag{16}$$

This δ function is a Fermi rather than Bose element, and it is an *odd* function

$$\begin{aligned}\delta(S) &= -\delta(-S), \\ \delta(0) &= 0.\end{aligned}\tag{17}$$

The last property is very convenient for computations of the SUSY graphs.

The generalized Θ function satisfies the usual property

$$\Theta(A, B) + \Theta(B, A) = 1,\tag{18}$$

from which it follows that $\Theta(A, A) = \frac{1}{2}$.

We shall also need the generalization of the finite range integral

$$\int_A^B dS \equiv \int_{-\infty}^{\infty} ds \int d\theta \Theta(A, S) \Theta(S, B),\tag{19}$$

as well as the multiple ordered integral

$$\int_A^B d^n S \equiv \int_{-\infty}^{\infty} ds_1 \int d\theta_1 \dots \int_{-\infty}^{\infty} ds_n \int d\theta_n \Theta(A, S_1) \Theta(S_1, S_2) \dots \Theta(S_{n-1}, S_n) \Theta(S_n, B). \tag{20}$$

The following superfield differential forms will play an important role

$$\begin{aligned}dX_\mu(S) &= ds d\theta DX_\mu(S), \\ d^n X_{\{\mu\}}(\{S\}) &= dX_{\mu_1}(S_1) \dots dX_{\mu_n}(S_n).\end{aligned}\tag{21}$$

By construction these forms are globally SUSY, later we prove that they are locally SUSY as well. We are using the curly brackets $\{S\}$ for lists of indexes or arguments and the tensor power notation

$$F_{\{\mu\}}^n(\{S\}) \equiv \prod_{i=1}^n F_{\mu_i}(S_i).\tag{22}$$

The superfield differential 1-form satisfies the usual integration identities

$$\begin{aligned}\int_A^B dF(S) &= F(B) - F(A), \\ d(F(S)G(S)) &= dF(S)G(S) + F(S)dG(S),\end{aligned}\tag{23}$$

which allow integration by parts. The finite range integrals are additive

$$\int_A^B dF(S)G(S) = \int_A^C dF(S)G(S) + \int_C^B dF(S)G(S).\tag{24}$$

The integrals of multiple forms satisfy the composition law

$$\int_A^B dP_\lambda(S) \int_A^S d^m P_{\{\mu\}}(\{U\}) \int_S^B d^n P_{\{\nu\}}(\{V\}) = \int_A^B d^{m+n+1} P_{\{\rho\}}(\{W\}),\tag{25}$$

with the joined list of indexes

$$\{\rho\} = \{\{\mu\}, \lambda, \{\nu\}\}.\tag{26}$$

as well as the multiplication laws, the simplest of which is

$$\int_A^B dP_\lambda(U) \int_A^B d^n P_{\{\mu\}}(\{V\}) = \sum_{l=0}^n \int_A^B d^{n+1} P_{\{\nu(l)\}}(\{S\}), \quad (27)$$

where

$$\{\nu(l)\} = \{\mu_1, \dots, \mu_l, \lambda, \mu_{l+1}, \dots, \mu_n\}. \quad (28)$$

In other words the index λ is inserted at the position $l + 1$ in the list of μ indexes.

Both of these laws follow from the local SUSY which we prove in the next Section and the supersymmetrization theorem which we prove in Appendix C.

Using these forms, we can define the SUSY path exponential with initial point $A = (a, \theta_a)$ and final point $B = (b, \theta_b)$

$$\mathcal{U}_A^B[A_\mu, X_\mu] \equiv \hat{T} \exp \left(\int_A^B dX_\mu(S) A_\mu(X(S)) \right) \equiv \sum_{n=0}^{\infty} \int_A^B d^n X_{\{\mu\}}(\{S\}) A_{\{\mu\}}^n(\{X(S)\}). \quad (29)$$

One may verify the multiplicativity of the path exponent

$$\mathcal{U}_A^B[A_\mu, X_\mu] = \mathcal{U}_A^S[A_\mu, X_\mu] \mathcal{U}_S^B[A_\mu, X_\mu], \quad (30)$$

its differential with respect to the endpoints

$$\begin{aligned} d\mathcal{U}_A^S[A_\mu, X_\mu] &= \delta(S - A) + \mathcal{U}_A^S[A_\mu, X_\mu] dX_\mu(S) A_\mu(X(S)), \\ d\mathcal{U}_S^B[A_\mu, X_\mu] &= -\delta(S - B) - dX_\mu(S) A_\mu(X(S)) \mathcal{U}_S^B[A_\mu, X_\mu], \end{aligned} \quad (31)$$

as well as the usual formula for its variation with respect to the gauge potential

$$\delta_{A_\mu} \mathcal{U}_A^B[A_\mu, X_\mu] = \int_A^B dX_\mu(S) \mathcal{U}_A^S[A_\mu, X_\mu] \delta A_\mu(X(S)) \mathcal{U}_S^B[A_\mu, X_\mu]. \quad (32)$$

In particular, for the gauge variation $\delta_{gauge} A_\mu(X) = \nabla_\mu \alpha(X)$ (note that the gauge parameter $\alpha(X)$ having the superfield $X(S)$ as its argument)

$$\delta_{gauge} \mathcal{U}_A^B[A_\mu, X_\mu] = \int_A^B dX_\mu(S) \mathcal{U}_A^S[A_\mu, X_\mu] \nabla_\mu \alpha(X(S)) \mathcal{U}_S^B[A_\mu, X_\mu]. \quad (33)$$

Using above endpoint derivatives we reduce the integral to the total derivative, up to the δ terms, which yields

$$\delta_{gauge} \mathcal{U}_A^B[A_\mu, X_\mu] = \mathcal{U}_A^B[A_\mu, X_\mu] \alpha(X(B)) - \alpha(X(A)) \mathcal{U}_A^B[A_\mu, X_\mu]. \quad (34)$$

We got something new here. The trace of this variation will vanish provided the superloop is closed, i.e. $X(B) = X(A)$. The Bose and Fermi parts by themselves could be opened. The gap in Bose part $x(b) - x(a) = -\theta_b \psi(b) + \theta_a \psi(a)$ represent the nilpotent even element of the Grassmann algebra. Its higher tensor products $(x(b) - x(a))_{\mu_1} \dots (x(b) - x(a))_{\mu_n}$ vanish at $n > 2$. This is less restrictive than the ordinary gauge invariance which would demand $n > 0$.

We are in a position to define the superloop

$$\mathcal{W}[X]_A^B = \frac{1}{N_c} \left\langle \text{tr} \mathcal{U}_A^B[A_\mu, X_\mu] \right\rangle_{A_\mu}. \quad (35)$$

In usual notation the SWLoop reads

$$\mathcal{W}[X]_0^1 = \frac{1}{N_c} \text{tr} \hat{T} \exp \left(\int_0^1 ds \left(A_\mu(x) x'_\mu + \frac{1}{2} \psi_\mu \psi_\nu F_{\mu\nu}(x) \right) \right) \equiv \mathcal{W}[x, \psi]. \quad (36)$$

This formula was first obtained in [12, 13] for the same purpose of supersymmetric description of spin $\frac{1}{2}$ particle. In case of the Grassmann variables at the end points A, B , there will be extra terms. Say, in Abelian case there will be extra factors $\exp(\pm \theta \psi_\mu A_\mu(x))$ at the endpoints.

The relation between this formula and the one with superfield is quite amazing. The $\psi\psi\partial A$ part of the $\psi\psi F$ terms comes from the expansion of the potential in the action, and the commutator terms $\psi\psi AA$ come from the $\theta_i \theta_j \delta(s_i - s_j)$ term in the SUSY definition of the the ordered product [11]. The SUSY and the non-Abelian gauge invariance work in close collaboration, which is quite common, as we shall see.

The vacuum energy can be written in terms of superfields as follows

$$\mathcal{E} = \frac{N_c}{2} \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) e^{-Tm_0^2} \int \mathcal{D}X \exp \left(-\frac{1}{4T} \int_0^1 dX_\mu(S) D^2 X_\mu(S) \right) \text{tr} \hat{T} \exp \left(\int_0^1 dX_\mu(S) B_\mu(X(S)) \right) \mathcal{W}[X]_0^1. \quad (37)$$

The ordinary perturbation theory corresponds to expansion in both flavor and color gauge fields and taking the Gaussian path integrals term by term. The role of the Grassmann part $\theta\psi_\mu(s)$ of the path $X_\mu(S)$ is to get correct numerators of the Feynman diagrams, coming from the Dirac traces in usual language.

This follows from the explicit form of the kinetic energy

$$\frac{1}{4T} \int_0^1 dX_\mu(S) D^2 X_\mu(S) = \frac{1}{4T} \int_0^1 ds \left(x'_\mu(s) x'_\mu(s) + \psi'_\mu(s) \psi_\mu(s) \right). \quad (38)$$

One could rescale $s \Rightarrow T s$, $\psi_\mu \Rightarrow \sqrt{T} \psi_\mu$, which only affects this term, as the rest of the terms are parametric invariant. Then the kinetic energy will take a familiar form, which corresponds to the usual free particle. The ψ part after quantization reproduces the usual Clifford algebra of Dirac matrices. The extra factors of mass in the local measure, coming from rescaling of ψ_μ , are just the correct factors needed to normalize the Wiener measure for $x_\mu(s)$.

We mentioned this only to make our path integral look more familiar. For our purposes this transformation is not needed, in fact, it would only make the formulas look more complex than they actually are.

For example, the Euler-Heisenberg effective Lagrangian [14] in homogeneous external field $B_\nu(x) = \frac{i}{2} B_{\mu\nu} x_\mu$ is computed in just few lines in Appendix B.

4 Correlation Functions of Vector Currents

Let us expand the vacuum energy in powers of the external field. The expansion coefficients are related to the correlation functions of the vector flavour currents.

We use the Fourier integrals

$$B_\mu(X) = \int \frac{d^d k}{(2\pi)^d} B_\mu(k) \exp(i k_\mu X_\mu), \quad (39)$$

which also define the SUSY extension to $X = X(S)$. In the n -th order we shall have the product of n exponentials, which can be written in terms of the supermomentum

$$\exp\left(i \sum_i k_\mu^i X_\mu(S_i)\right) = \exp\left(i \int_0^1 dK_\mu(S) X_\mu(S)\right) = \exp\left(-i \int_0^1 dX_\mu(S) K_\mu(S)\right), \quad (40)$$

with the supermomentum

$$\begin{aligned} K_\mu(S) &= \sum_i k_\mu^i \Theta(S_i, S), \\ dK_\mu(S) &= \sum_i k_\mu^i dS \delta(S - S_i). \end{aligned} \quad (41)$$

We need the standard SUSY functional derivative for a superfield $P_\mu(s) = p_\mu(s) + \theta \varphi_\mu(s)$

$$\frac{\delta}{\delta P_\mu(S)} = \frac{\delta}{\delta \varphi_\mu(s)} + \theta \frac{\delta}{\delta p_\mu(s)}, \quad (42)$$

which satisfies the identity

$$\frac{\delta}{\delta P_\mu(S)} P_\nu(S') = \delta_{\mu\nu} \delta(S - S'). \quad (43)$$

In the same way, as we introduced the superfield 1-form $dX_\mu(S)$ we could introduce an invariant superfield gradient 1-form

$$\frac{\partial}{\partial P_\mu(S)} \equiv dS \frac{\delta}{\delta P_\mu(S)}. \quad (44)$$

The motivation for this notation is the following identity

$$\frac{\partial}{\partial P_\mu(S)} \int_A^B dX_\nu(U) P_\nu(U) = dX_\nu(S), \quad (45)$$

which means that this gradient form acts on the integral of differential form just as it does in Bosonic case.

The vacuum energy takes the form

$$\mathcal{E} = \sum_{n=0}^{\infty} i^n \int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_n}{(2\pi)^d} \delta^d\left(\sum_i k_i\right) B_{\mu_1}^{a_1}(k_1) \cdots B_{\mu_n}^{a_n}(k_n) \left\langle J_{\mu_1}^{a_1} \cdots J_{\mu_n}^{a_n} \right\rangle(k_1 \cdots k_n), \quad (46)$$

with the following planar connected correlation functions of flavour currents

$$\begin{aligned} \left\langle J_{\mu_1}^{a_1} \dots J_{\mu_n}^{a_n} \right\rangle (k_1 \dots k_n) = \\ \frac{N_c}{2} \text{tr} (\tau_{a_1} \dots \tau_{a_n}) Z \left(\frac{1}{4} \int_0^1 \frac{\partial}{\partial K_\mu(S)} D \frac{\delta}{\delta K_\mu(S)} \right) \int_0^1 \frac{\partial \mathcal{M}[K]_0^1}{\partial K_{\mu_1}(S_1) \dots \partial K_{\mu_n}(S_n)}, \end{aligned} \quad (47)$$

where we introduced the following operator function

$$Z(\hat{L}) = \lim_{\alpha \rightarrow 0} \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) \exp \left(-T m_0^2 - \frac{\hat{L}}{T} \right). \quad (48)$$

This is a certain Bessel function of its operator argument.

The loop functional

$$\mathcal{M}[P]_A^B = \int \mathcal{D}X \delta^d(X(A)) \exp \left(\imath \int_A^B dP_\mu(S) X_\mu(S) \right) \mathcal{W}[X]_A^B; \quad (49)$$

is a Fourier transform in superspace.

5 Chiral Anomaly

There is a simpler observable, which is not influenced by quark kinetic energy. This is the chiral anomaly

$$I[B] = \left\langle \text{tr} \gamma_5 \exp \left(-T (\imath \gamma_\mu \nabla_\mu)^2 \right) \right\rangle. \quad (50)$$

as a functional of the external flavour field B . The famous index theorem states that it is T -independent: due to the γ_5 symmetry of the Dirac operator, all the finite eigenvalues drop from the trace, so that only the T -independent contribution of the zero eigenvalues remain. In absence of quantum fields, this allows to compute it, taking the limit $T \rightarrow 0$ and using the WKB expansion. This leads to the Grassmann integral over the zero mode

$$I_{cl}[B] \propto V \int d^d \psi^0 \text{tr} \hat{T} \exp \left(\frac{1}{2} B_{\mu\nu} \psi_\mu^0 \psi_\nu^0 \right) \propto V \varepsilon_{\mu_1 \dots \mu_d} \text{tr} B_{\mu_1 \mu_2} \dots B_{\mu_{d-1} \mu_d}. \quad (51)$$

In case of abelian field this is the pfaffian $\sqrt{\det \imath B}$.

The quantum chiral anomaly, $I[B]$ according to the same index theorem, is simply the vacuum average of the classical chiral anomaly of the direct sum of the two gauge fields $\langle I_{cl}[B \oplus A] \rangle$ computed in absence of the B field.

$$I[B] = \langle I_{cl}[B \oplus A] \rangle = N_c I_{cl}[B] + N_f \langle I_{cl}[A] \rangle. \quad (52)$$

The last term vanishes by parity, so that the chiral anomaly stays the same as in the free quark theory.

Let us now relate this vacuum average to our momentum loops. We have to repeat the steps of the previous section, with two changes. First, we choose the different boundary

conditions. Now both $x(s)$ and $\psi(s)$ are periodic, to compensate for the γ_5 . After that we can drop the kinetic energy term, according to the index theorem.

We find the following formula

$$N_c I_{cl}[B] = \frac{N_c}{2} \int_{\text{periodic}} \mathcal{D}X \hat{T} \exp \left(\int_0^1 dX_\mu(S) B_\mu(X(S)) \right) \mathcal{W}[X], \quad (53)$$

which provides an important normalization condition for the SMloop considered below.

In particular, consider the case of constant non-abelian field $B_\mu(X) = \text{const.}$ This corresponds to constant field strength $B_{\mu\nu} = [B_\mu, B_\nu]$. The chiral anomaly reads

$$I_{cl}[B] \propto V \varepsilon_{\mu_1 \dots \mu_d} \text{tr} B_{\mu_1 \mu_2} \dots B_{\mu_{d-1} \mu_d} \propto V \varepsilon_{\mu_1 \dots \mu_d} \text{tr} B_{\mu_1} B_{\mu_2} \dots B_{\mu_d}. \quad (54)$$

Comparing this with above path integral we get the sum rule

$$\int_{\text{periodic}} \mathcal{D}X \int_0^1 d^d \hat{X}_{\{\mu\}}(\{S\}) \mathcal{W}[X] \propto \varepsilon_{\mu_1 \dots \mu_d}. \quad (55)$$

This fixes the coefficients in front of $K_{\mu_1} \dots K_{\mu_d}$ in Fourier transform of $\mathcal{W}[X]$ with periodic boundary conditions.

6 Local SUSY

Let us study the symmetries of the SWloop. It is invariant under parametric transformations

$$\delta x_\mu(s) = \alpha(s) x'_\mu(s), \quad \delta \psi_\mu(s) = \alpha(s) \psi'_\mu(s) + \frac{1}{2} \alpha'(s) \psi_\mu(s), \quad (56)$$

which is straightforward to verify. In addition it is invariant under local SUSY transformations

$$\delta x_\mu(s) = \beta(s) \psi_\mu(s), \quad \delta \psi_\mu(s) = \beta(s) x'_\mu(s). \quad (57)$$

with some Grassmann variable $\beta(s)$. Together these transformations define the reparametrization of the superfield,

$$\delta X_\mu = X_\mu(s + \delta s, \theta + \delta \theta) - X_\mu(s, \theta) = \left(\delta s \frac{\partial}{\partial s} + \delta \theta \frac{\partial}{\partial \theta} \right) X_\mu(s, \theta), \quad (58)$$

with

$$\delta s = \alpha(s) + \theta \beta(s), \quad \delta \theta = \frac{1}{2} \alpha'(s) \theta + \beta(s). \quad (59)$$

From now on, we shall always mean the local rather than the global SUSY. In Abelian theory this is just the formal symmetry of the gauge field part of the action. One may verify that the variation of the A - term in a Hamiltonian after integration by parts cancels the variation of the F - term. In the last variation the extra terms with gradients of $F_{\mu\nu}$ add up to zero by virtue of the Bianchi identity

$$\psi_\mu \psi_\nu \psi_\lambda \partial_\lambda F_{\mu\nu} = 0 \quad (60)$$

In the non-Abelian theory the same can be proven for the the ordered exponent. In the superfield language the SUSY of the action follows from the opposite transformation laws of the integration measure and covariant derivative

$$\delta dS = \left(\frac{\partial(\delta s, \delta\theta)}{\partial(s, \theta)} - 1 \right) dS = \left(\frac{\partial\delta s}{\partial s} + \frac{\partial\delta\theta}{\partial\theta} \right) dS = \left(\frac{1}{2}\alpha'(s) + \theta\beta'(s) \right) dS, \quad (61)$$

and

$$\delta D = \delta \frac{\partial}{\partial\theta} + \delta\theta \frac{\partial}{\partial s} + \theta\delta \frac{\partial}{\partial s} = - \left(\frac{1}{2}\alpha'(s) + \theta\beta'(s) \right) D. \quad (62)$$

so that the differential $d = dSD$ stays invariant.

It remains to check the Θ function (15). The parametric invariance is obvious, as for the local SUSY, it can be verified as follows

$$\begin{aligned} \delta\Theta(S, S') &= \delta L(S, S')\delta(s - s' + \theta\theta') = \\ &= (\theta + \theta')(\beta(s') - \beta(s))\delta(s - s' + \theta\theta') = \\ &= (\theta + \theta')(\beta(s') - \beta(s))\delta(s - s') = 0. \end{aligned} \quad (63)$$

It is interesting that Andreev and Tseytlin [11], who introduced this SUSY Θ function, did not notice its local SUSY, as they were working in a superstring theory, where this local symmetry was broken down to a global one.

From the point of view of the 1D supergravity this local SUSY corresponds to the fact that the Wilson loop does not depend on the metric in 1D superspace, described by the einbein-gravitino superfield. Thus it measures topology in this space, which is rather simple.

It is quite instructive to verify the local SUSY directly, using the components of the superfield. We refer the reader to the review paper [3] where the loop kinematics and dynamics was discussed in great detail. Using the methods of that paper one can prove the SUSY of the non-Abelian SWLoop as follows.

From the point of view of loop dynamics the SWLoop is not an independent functional. It relates to the Wilson loop by the linear operator

$$\mathcal{W}[x, \psi]_0^1 = \hat{T} \exp \left(\int_0^1 ds \frac{1}{2} \psi_\mu \psi_\nu \frac{\partial}{\partial \sigma_{\mu\nu}} \right) W[x]_0^1. \quad (64)$$

Using this representation, the functional derivatives can be readily computed

$$\begin{aligned} \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta \psi_\mu(s)} &= \psi_\nu(s) \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta \sigma_{\mu\nu}}(s), \\ \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta x_\mu(s)} &= \left(x'_\beta(s) \delta_{\alpha\mu} + \frac{1}{2} \psi_\alpha(s) \psi_\beta(s) \partial_\mu(s) \right) \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta \sigma_{\alpha\beta}}(s). \end{aligned} \quad (65)$$

It is implied that the left functional derivative is used for the Grassmann variable ψ .

Multiplying the last relation by $\psi_\mu(s)$ and using the non-Abelian Bianchi identity [3], we get

$$\psi_\mu(s) \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta x_\mu(s)} = x'_\nu(s) \psi_\mu(s) \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta \sigma_{\mu\nu}}(s) = -x'_\nu(s) \frac{\delta \mathcal{W}[x, \psi]_0^1}{\delta \psi_\nu}(s), \quad (66)$$

which is a differential form of the SUSY relation.

This local supersymmetry is a genuine symmetry of the loop dynamics in the pseudoscalar sector. It does not require any external einbein-gravitino fields, which were needed to supersymmetrize the massive spin $\frac{1}{2}$ particle. In other words, we are dealing with topological SUSY Quantum Mechanics.

7 SuperLoop Equation

The SWLoop equation is obtained as a SUSY extension¹ of the usual loop equation [2, 3]. It looks exactly the same as the ordinary loop equation, with a superfield $X_\mu(S)$ in place of the ordinary coordinate $x_\mu(s)$,

$$0 = \int_0^1 dS \left(\frac{\delta \mathcal{W}[X]_0^1}{\delta X_\mu(0) \delta X_\mu(S)} + N_c g_0^2 DX_\mu(0) DX_\mu(S) \delta^d(X(S) - X(0)) \mathcal{W}[X]_0^S \mathcal{W}[X]_S^1 \right). \quad (67)$$

In Appendix C we derive the supersymmetrization theorem, which states that any locally SUSY functional of the curve $\Gamma : S = (s(\tau), \theta(\tau))$ which vanishes identically for Γ being the real axis: $\theta(\tau) \equiv 0$ must vanish identically in the whole superspace. The bottom line is, the local SUSY transformation has one Bose and one Fermi function of one variable, so it allows us to move and reparametrize any curve from the real axis anywhere in superspace.

This resembles the analytic continuation from the real axis to the complex plane. As long as the Cauchy-Riemann equations are satisfied this analytic continuation looks like a trivial replacement of the real number by a complex number in any equation. The hidden cost is just the Cauchy-Riemann equations which should be checked in every case.

One may think that the SUSY loop equation follows from the ordinary loop equation by means of this naive SUSY extension, but this is not true. The supersymmetrization theorem does not literally apply here. The two arbitrary functions of the SUSY transformation are not sufficient to eliminate the Grassmann components of d -vector superfield $X_\mu(S)$. One has to rederive the equation from scratch, using the superfields instead of coordinates.

Let us examine the linear term in components

$$\int_0^1 ds \frac{\delta \mathcal{W}[X]_0^1}{\delta \psi_\mu(0) \delta x_\mu(s)} = -\partial_\mu(0) \frac{\delta}{\delta \psi_\mu(0)} \mathcal{W}[X]_0^1 = -\psi_\nu(0) \nabla_\mu F_{\mu\nu}(x(0)) \otimes \mathcal{W}[X]_0^1. \quad (68)$$

with the colour matrix $\nabla_\mu F_{\mu\nu}(x(0))$ being inserted in the beginning of the the ordered product.

According to the quantum YM equations, this can be replaced by

$$-\psi_\nu(0) g_0^2 \frac{\delta}{\delta A_\nu(x(0))} \otimes \mathcal{W}[X]_0^1 \quad (69)$$

with the same agreement about insertion inside the the ordered product. The variation of the path the ordered exponential goes as follows (each step can be proven using the above Taylor

¹I am indebted to Sasha Polyakov for the advice to switch to superfields here.

expansion with SUSY Θ functions)

$$\begin{aligned} \frac{\delta}{\delta A_\nu(x(0))} \otimes \hat{T} \exp \left(\int_0^1 dX_\mu(S) A_\mu(X(S)) \right) = \\ \int_0^1 dX_\mu(S) \delta^d(X(S) - x(0)) \\ \hat{T} \exp \left(\int_0^S dX_\mu(U) A_\mu(X(U)) \right) \otimes \hat{T} \exp \left(\int_S^1 dX_\mu(V) A_\mu(X(V)) \right). \end{aligned} \quad (70)$$

The δ^d function here is the usual delta function in d dimensional Euclidean space, with argument replaced by the superfield. It should be understood in the sense of a Fourier integral

$$\begin{aligned} A_\mu(X(S)) &= \int \frac{d^d k}{(2\pi)^d} A_\mu(k) \exp(\imath k_\mu X_\mu(S)), \\ \frac{\delta A_\mu(X(S))}{\delta A_\nu(x(0))} &= \int d^d q \exp(-\imath q_\mu x_\mu(0)) \frac{\delta A_\mu(X(S))}{\delta A_\nu(q)} = \\ &\int \frac{d^d k}{(2\pi)^d} \int d^d q \delta^d(k - q) \exp(\imath k_\mu X_\mu(S)) \exp(-\imath q_\mu x_\mu(0)) = \\ &\int \frac{d^d k}{(2\pi)^d} \exp(\imath k_\mu (X_\mu(S) - x_\mu(0))) \equiv \delta^d(X(S) - x(0)). \end{aligned} \quad (71)$$

Recalling that $\psi_\nu(0) = DX_\nu(0)$ and $x_\nu(0) = X_\nu(0)$ we get the above superloop equation.

Let us see how this works in components. The integral over the θ variable in $\int_0^1 dX_\mu(S)$ picks up the first derivative in the superfield expansion. This is equivalent to applying the operator $\psi_\mu(s)\partial_\mu(s)$ to the $\delta^d(X(S) - X(0))$ function as well as to the SWLoops $\mathcal{W}[X]_0^S \mathcal{W}[X]_S^1$. In the latter case these are the covariant derivatives $\partial_\mu(s)$.

Like in a SWLoop itself, the various parts of the covariant derivative come from various places, supported by local SUSY. The $\frac{\partial}{\partial x}$ terms in covariant derivatives come from expansion of the X superfield, and the A_μ terms come from the expansions of the Θ functions in the ordered product (the last Θ function $\Theta(S_n, S)$ in the $\mathcal{W}[X]_{0S}$ and the first one $\Theta(S, S_1)$ in the $\mathcal{W}[X]_{S1}$ are differentiated). We get simply

$$x'_\mu(s) + \psi_\nu(s)\psi_\mu(s)\partial_\nu(s). \quad (72)$$

as an operator applied to the rest of the factors.

The same expression could be obtained by an honest variation of the SWloop. In superfields everything looks so simple, but in fact all the subtleties of the loop dynamics are automatically taken care of.

Let us discuss the boundary conditions. The initial loop $\mathcal{W}[X]_0^1$ was periodic in $x(s)$ but the boundary values of $\psi(s)$ were left free. The loops $\mathcal{W}[X]_0^S, \mathcal{W}[X]_S^1$ in the nonlinear part of equation belong to the same class.

The right boundary value $\psi(s)$ of the first part of the path coincides with the left boundary value for the second part, so that we have a convolution in the loop equation.

The physical loops, which enter the vacuum energy and the vacuum topological charge, are closed in ψ with antiperiodic or periodic boundary conditions. This can be achieved by setting $\psi(1) = \pm\psi(0)$ and integrating over $\psi(0)$.

8 Momentum Superloops

Let us now go to supermomentum space. There is one important point to clarify before we do it. The natural definition would imply the periodic momentum $P_\mu(S)$, $P_\mu(1) = P_\mu(0)$. However, the loop equation does not close in the space of periodic momentum loops.

This issue was discussed before [8, 1]. In general, one could introduce some gaps $\Delta P(S_i) = P(S_i+0) - P(S_i-0)$, in which case the momentum loop equation relates the n -gap functional to bilinear superposition of $l+1$ and $n-l+1$ functionals, with $l = 0, \dots, n$.

These gaps correspond to slow decrease $\frac{1}{n}$ of the Fourier harmonics of $p_\mu(s)$, which is an unpleasant complication of momentum loop dynamics. As we see it now, this complication can be avoided. Take *non-periodic* functions, $P_\mu(1) \neq P_\mu(0)$ which are otherwise smooth. The derivative $DP_\mu(S)$ could be finite everywhere including the endpoints. However, the endpoint derivatives are different $DP_\mu(0+) \neq DP_\mu(1-)$.

From the point of view of the general framework, with gaps, this corresponds to only one gap, between 1 and 0. The inside momentum loops will also have one gap each, between *their* initial and finite points respectively $S, 0$ and $1, S$. In other words, the "loop" has a topology of an interval, rather than a circle. Cutting the interval at some inside point we get two intervals, so that the topology is trivially preserved.

The Fourier transformation is defined as follows

$$\mathcal{M}[P, \psi, \psi']_A^B = \int \mathcal{D}X \exp \left(i \int_A^B X_\mu(S) dP_\mu(S) \right) \mathcal{W}[X], \quad (73)$$

where ψ, ψ' are boundary values of the Fermi part of the path $X_\mu(S)$.

After the Fourier transformation the loop equation reads

$$DP_\mu(0) \mathcal{M}[P, \psi, \psi']_0^1 \int_0^1 dP_\mu(S) = -N_c g_0^2 \int_0^1 dS \int d^d \psi(s) \frac{\delta \mathcal{M}[P, \psi, \psi(s)]_0^S \mathcal{M}[P, \psi(s), \psi']_S^1}{\delta P_\mu(0) \delta P_\mu(S)}. \quad (74)$$

This equation is not as simple as it looks. Several things need to be clarified.

The coordinate δ function disappeared in the same way as in the ordinary momentum loop equation [8]. The product $\delta^d(X(0))\delta^d(X(S) - X(0))$ can be replaced by $\delta^d(X(0))\delta^d(X(S))$ which is just what is needed to get the correct integrand in the product of the two Fourier transforms. The $\delta^d(X(0))$ function eliminates translations in $\mathcal{W}[X]_0^S$ and the other one, $\delta^d(X(S))$ does the same for $\mathcal{M}[X]_S^1$.

Note that in the last delta function the θ term in superfield is present, which adds the terms with gradients of the delta function. This is not a problem, since the same SUSY δ function was included in the definition of the Fourier transform. We had to do it to preserve

the local SUSY. By virtue of SUSY the choice of the point to fix on a loop is arbitrary, so that we could replace $\delta^d(X(S))$ by $\delta^d(X(1))$ in the Fourier integral for $\mathcal{M}[P]_S^1$.

Some subtleties of the factorization of the Fourier integrals are discussed in Appendix D.

Note, that there was no delta function for the Fermi part of the loop, therefore the integration over $\psi(s)$ was not eliminated. There is no explicit dependence of these values in our equation, just the integration. Therefore, we can look for the diagonal solution

$$\mathcal{M}[P, \psi, \psi']_A^B = \delta^d\left(\frac{1}{2}(\psi + \psi')\right) \mathcal{M}[P+]_A^B + \delta^d\left(\frac{1}{2}(\psi - \psi')\right) \mathcal{M}[P-]_A^B. \quad (75)$$

Here $\mathcal{M}[P\pm]$ correspond to the two types of boundary conditions

$$\mathcal{M}[P\pm]_A^B = \int d^d\psi \mathcal{M}[P, \psi, \pm\psi]_A^B. \quad (76)$$

We used the fact that $\delta^d(0) = 0$ for the Fermi variables.

Thus $\mathcal{M}[P+]$ describes the chiral anomaly, and $\mathcal{M}[P-]$ - the vacuum energy (in the latter case one should also include the quark kinetic energy – see below).

Then, the ψ delta functions integrate out, and we are left with the single equation

$$DP_\mu(0)\mathcal{C}[P]_0^1 \int_0^1 dP_\mu(S) = 2N_c g_0^2 \int_0^1 dS \frac{\delta\mathcal{C}[P]_0^S}{\delta P_\mu(0)} \mathcal{C}[P]_S^1, \quad (77)$$

where we introduced the chiral functional

$$\mathcal{C}[P] = \frac{1}{2} (\mathcal{M}[P-] + \mathcal{M}[P+]) \quad (78)$$

which corresponds to projection operator $\frac{1}{2}(1 + \gamma_5)$ inserted into the Dirac matrix trace. This projection operator commutes with the Lorentz generators

$$\sigma_{\mu\nu} = \frac{1}{4i} [\gamma_\mu, \gamma_\nu], \quad (79)$$

which enter the Pauli term $\sigma_{\mu\nu} F_{\mu\nu}$ in the square of the Dirac operator. Hence, the chiral functional corresponds to the chiral generators $\frac{1}{2}(1 + \gamma_5)\sigma_{\mu\nu}$. The usual even/odd functionals correspond to the even/odd parity part of the chiral functional.

In spite of chiral loop equation being parity even, we have to include the terms $\varepsilon_{\mu_1 \dots \mu_d}$ in the coefficients. These terms are fixed by the chiral anomaly, as discussed above. We shall come back to this issue below, after we introduce external field.

The SMLoop equation can be simplified if we introduce the SUSY area derivative

$$\frac{\delta\mathcal{C}[P]_A^B}{\delta P_\mu(S)} = DP_\nu(S) \frac{\partial\mathcal{C}[P]_A^B}{\partial\Sigma_{\mu\nu}(S)}, \quad (80)$$

or, in terms of differential forms

$$\frac{\partial\mathcal{C}[P]_A^B}{\partial P_\mu(S)} = dP_\nu(S) \frac{\partial\mathcal{C}[P]_A^B}{\partial\Sigma_{\mu\nu}(S)}, \quad (81)$$

Later we compute the area derivatives for the terms of the SUSY Taylor expansion of the SMLoop. As discussed before, we assume the smooth non-periodic function $P(U)$, so that $P_\mu(S+) = P_\mu(S-)$ but $P_\mu(1) \neq P_\mu(0)$.

In the SMLoop equation both linear and bilinear terms involve the $dP_\mu(0) = \frac{1}{2}dP_\mu(1-) + \frac{1}{2}dP_\mu(0+)$ form. *Both* of these forms being arbitrary, we could "cancel" them, i.e. leave only the tensor coefficients multiplying each form.

This yields *two* equations

$$0 = \int_0^1 dP_\mu(S) \left(\delta_{\lambda\mu} \mathcal{C}[P]_0^1 + N_c g_0^2 \frac{\partial \mathcal{C}[P]_0^S \mathcal{C}[P]_S^1}{\partial \Sigma_{\mu\nu}(\xi) \partial \Sigma_{\lambda\nu}(S)} \right), \quad (82)$$

where $\xi = 0$ or $\xi = 1$.

9 The External Field

The problem with the momentum loop dynamics is that it is non-perturbative. The zeroth approximation of perturbation theory corresponds to singular momentum loop which singularity we expect to disappear in the full solution. The only way we can study this full solution so far is to expand in powers of momenta. The expansion coefficients are expected to be proportional to the powers of confinement radius which is infinite in perturbation theory.

The loop equation being nonlinear, we can never know whether the expansion we build, really matches the perturbative QCD as it should at confinement scale of momenta. What if this is a wrong solution, some strong coupling artifact, like the ones in lattice gauge theory?

There is a following way around this obstacle. Add the external Abelian constant field $B_{\mu\nu}$, then at large $B_{\mu\nu}$ the asymptotic freedom will hold, and we will know the solution. At any finite field the SMLoop will be still expandable in powers of momenta.

The B -dependence of expansion coefficients can be found from the following identities

$$\frac{\partial \mathcal{C}[P]_A^B}{\partial B_{\mu\nu}} = \frac{i}{2} \int_A^B dS_1 dS_2 \Theta(S_1, S_2) \frac{\delta \mathcal{C}[P]_A^B}{\delta P_\mu(S_1) \delta P_\nu(S_2)}. \quad (83)$$

These identities follow directly from the original functional integral, with the Abelian external field factor inserted

$$\hat{T} \exp \left(i \int_A^B dX_\mu(S) B_\mu(X(S)) \right) = \exp \left(\frac{i}{2} B_{\mu\nu} \int_A^B \Theta(S_1, S_2) dX_\mu(S_1) dX_\nu(S_2) \right). \quad (84)$$

In the Fourier integral one could replace $dX_\mu(S)$ by $i \frac{\delta}{\delta P_\mu(S)}$.

It is convenient to introduce the area derivatives, then one finds in (83) after some algebra

$$\begin{aligned} \frac{\partial \mathcal{C}[P]_A^B}{\partial B_{\mu\nu}} &= \frac{i}{4} \left(\frac{\partial}{\partial \Sigma_{\mu\nu}(A)} + \frac{\partial}{\partial \Sigma_{\mu\nu}(B)} \right) \mathcal{C}[P]_A^B + \\ &\frac{i}{2} \int_A^B dP_\lambda(S_1) dP_\delta(S_2) \Theta(S_1, S_2) \frac{\partial \mathcal{C}[P]_A^B}{\partial \Sigma_{\mu\lambda}(S_1) \partial \Sigma_{\nu\delta}(S_2)}. \end{aligned} \quad (85)$$

As for the loop equations, they are modified in an obvious way, by adding the derivatives of this exponential to the operators $\frac{\delta}{\delta X_\mu(S)}$.

The resulting equations read

$$0 = \int_0^1 dP_\mu(S) \left(\left(\delta_{\rho\lambda} + \imath B_{\rho\sigma} \frac{\partial}{\partial \Sigma_{\sigma\lambda}} \right) \left(\delta_{\rho\mu} + \imath B_{\rho\delta} \frac{\partial}{\partial \Sigma_{\delta\mu}} \right) \mathcal{C}[P]_0^1 + \right. \\ \left. + N_c g_0^2 \frac{\partial \mathcal{C}[P]_0^S \mathcal{C}[P]_S^1}{\partial \Sigma_{\mu\nu}(\xi) \partial \Sigma_{\lambda\nu}(S)} \right), \quad (86)$$

where $\xi = 0$ or $\xi = 1$.

It is convenient to pull out of the SMLoop the exponential, corresponding to the free quark theory

$$\mathcal{C}[P]_A^B \Rightarrow \exp \left(\frac{\imath}{2} B_{\mu\nu}^{-1} \int_A^B P_\mu(S) dP_\nu(S) \right) \mathcal{C}[P]_A^B. \quad (87)$$

This factor being multiplicative around the loop, it will cancel on both sides of the loop equation. The only result will be the shift

$$\frac{\delta}{\delta P_\mu(S)} \Rightarrow \frac{\delta}{\delta P_\mu(S)} + \imath B_{\mu\nu}^{-1} D P_\nu(S). \quad (88)$$

This shift eliminates the DP term in the linear part of the momentum loop equation, and we get

$$B_{\nu\sigma} B_{\nu\delta} \int_0^1 dP_\mu(S) \frac{\partial}{\partial \Sigma_{\sigma\lambda}} \frac{\partial}{\partial \Sigma_{\delta\mu}} \mathcal{C}[P]_0^1 = \\ N_c g_0^2 \int_0^1 dP_\mu(S) \left(\imath B_{\mu\nu}^{-1} + \frac{\partial}{\partial \Sigma_{\mu\nu}(\xi)} \right) \left(\imath B_{\lambda\nu}^{-1} + \frac{\partial}{\partial \Sigma_{\lambda\nu}(\xi)} \right) \mathcal{C}[P]_0^S \mathcal{C}[P]_S^1, \quad (89)$$

Now, the perturbative solution would correspond to iterations of this equation in g_0 , starting with the free quark loop. The free quark computations are presented in Appendix C (see also below). The result for initial definition of chiral supermomentum loop functional reads

$$\mathcal{C}[P]_A^B \rightarrow \sqrt{\det \imath B} \theta \left(\sqrt{\det \imath B} \right) \exp \left(\frac{\imath}{2} B_{\mu\nu}^{-1} \int_A^B P_\mu(S) dP_\nu(S) \right) + O(g_0^2). \quad (90)$$

It is an interesting challenge to verify how the higher terms of perturbation theory in a background field are reproduced.

10 The Position Operator and Nonlinear Symmetry

Let us proceed with the momentum superloops. The next step in [1] was the introduction of the position operator \hat{X}_μ as a connection in momentum space.

The corresponding generalization of this Ansatz would be

$$\mathcal{C}[P]_A^B = \frac{1}{N_c g_0^2} \left\langle 0 \left| \hat{T} \exp \left(\imath \int_A^B dP_\mu(S) \hat{X}_\mu \right) \right| 0 \right\rangle. \quad (91)$$

The operator \hat{X}_μ is an ordinary Bose operator. There are no restrictions on the commutators $[\hat{X}_\mu, \hat{X}_\nu]$, so that these operators form a free algebra.

This Ansatz involves the superloop in momentum space with constant non-Abelian connection $\imath \hat{X}$. In components:

$$\hat{T} \exp \left(\imath \int_0^1 dP_\mu(S) \hat{X}_\mu \right) = \hat{T} \exp \left(\int_0^1 ds \left(\imath \hat{X}_\mu p'_\mu - \frac{1}{2} [\hat{X}_\mu, \hat{X}_\nu] \varphi_\mu \varphi_\nu \right) \right). \quad (92)$$

The above definition of the position operator \hat{X} does not assume periodicity of the superloop. One could translate \hat{X} by a constant vector operator \hat{C}_μ , which commutes with \hat{X}_μ as well as itself:

$$\begin{aligned} \hat{X}_\mu &\Rightarrow \hat{X}_\mu + \hat{C}_\mu, \\ [\hat{C}_\mu, \hat{C}_\nu] &= [\hat{C}_\mu, \hat{X}_\nu] = 0. \end{aligned} \quad (93)$$

As we shall see in a moment, the loop equation stays invariant.

This gauge freedom produces some arbitrary terms in the expectation values of products of \hat{X} operators, which, however, should all cancel out in the momentum superloops and other observables, by virtue of periodicity of the physical supermomentum.

Let us now compute the SUSY area derivative in momentum loop space. The result is a SUSY extension of the Mandelstam formula for the gauge potential $\imath \hat{X}_\mu$

$$\begin{aligned} N_c g_0^2 \frac{\partial \mathcal{C} [P]_A^B}{\partial \Sigma_{\mu\nu}(S)} = \\ - \left\langle 0 \left| \hat{T} \exp \left(\imath \int_A^S dP_\mu(U) \hat{X}_\mu \right) [\hat{X}_\mu, \hat{X}_\nu] \hat{T} \exp \left(\imath \int_S^B dP_\mu(V) \hat{X}_\mu \right) \right| 0 \right\rangle. \end{aligned} \quad (94)$$

In components it follows from the direct variation. In a superfield language it is even simpler. Applying the $\frac{\partial}{\partial P_\mu(S)}$ operator we get sum of terms with

$$\hat{X}_\mu \int_{S_{l-1}}^{S_{l+1}} d\delta(S_l - S) = \hat{X}_\mu (\delta(S_{l+1} - S) - \delta(S_{l-1} - S)). \quad (95)$$

The first term eliminates the integration over S_{l+1} , which yields $DP_\nu(S) \hat{X}_\mu \hat{X}_\nu$, the second one yields $-DP_\nu(S) \hat{X}_\nu \hat{X}_\mu$. The rest of terms add up to the product of two ordered exponentials.

Now we are in a position to study the superloop equations (82). The $\xi = 0$ equation becomes

$$\begin{aligned} \int_0^1 dP_\mu(S) \left(\delta_{\mu\lambda} \left\langle 0 \left| \hat{T} \exp \left(\imath \int_0^1 dP_\mu(U) \hat{X}_\mu \right) \right| 0 \right\rangle + \right. \\ \left. \left\langle 0 \left| \hat{T} \exp \left(\imath \int_0^S dP_\mu(V) \hat{X}_\mu \right) [\hat{X}_\mu, \hat{X}_\nu] \right| 0 \right\rangle \right. \\ \left. \left\langle 0 \left| [\hat{X}_\lambda, \hat{X}_\nu] \hat{T} \exp \left(\imath \int_S^1 dP_\mu(W) \hat{X}_\mu \right) \right| 0 \right\rangle + \right. \end{aligned} \quad (96)$$

$$\left\langle 0 \left| \hat{T} \exp \left(\imath \int_0^S dP_\mu(V) \hat{X}_\mu \right) [\hat{X}_\mu, \hat{X}_\nu] [\hat{X}_\lambda, \hat{X}_\nu] \right| 0 \right\rangle \\ \left\langle 0 \left| \hat{T} \exp \left(\imath \int_S^1 dP_\mu(W) \hat{X}_\mu \right) \right| 0 \right\rangle = 0.$$

Using the factorization of the ordered exponent we can write it as

$$\left\langle 0 \left| \int_0^1 dP_\mu(S) \hat{T} \exp \left(\imath \int_0^S dP_\mu(V) \hat{X}_\mu \right) \hat{H}(\hat{X})_{\mu\lambda} \hat{T} \exp \left(\imath \int_S^1 dP_\mu(W) \hat{X}_\mu \right) \right| 0 \right\rangle = 0, \quad (97)$$

where

$$\hat{H}(\hat{X})_{\mu\lambda} = \delta_{\mu\lambda} + [\hat{X}_\mu, \hat{X}_\nu] \left\{ |0\rangle \langle 0|, [\hat{X}_\lambda, \hat{X}_\nu] \right\}, \quad (98)$$

is some operator, related to \hat{X} .

But this is the symmetry property! The transformation

$$\delta \hat{X}_\mu = \delta \epsilon_\lambda \hat{H}(\hat{X})_{\mu\lambda}, \quad (99)$$

will result in the same variation of our Ansatz. Therefore, we conclude that the momentum loops $\mathcal{C}[P]$ are invariant with respect to these transformations with arbitrary infinitesimal vector $\delta \epsilon_\mu$. We could consider the family of operators $\hat{X}_\mu(t)$ parameterized by arbitrary path $\epsilon(t)$ and obeying the equations

$$\partial_t \hat{X}_\mu = \partial_t \epsilon_\lambda \hat{H}(\hat{X})_{\mu\lambda}. \quad (100)$$

The correlation functions must be independent of $\epsilon(t)$. The derivative

$$N_c g_0^2 \frac{\partial \mathcal{C}[P]_0^1}{\partial t} = \quad (101) \\ \imath \int_0^1 dP_\mu(S) \left\langle 0 \left| \hat{T} \exp \left(\imath \int_0^S dP_\mu(V) \hat{X}_\mu \right) \hat{H}(\hat{X})_{\mu\lambda} \partial_t \epsilon_\lambda \hat{T} \exp \left(\imath \int_S^1 dP_\mu(W) \hat{X}_\mu \right) \right| 0 \right\rangle = 0,$$

for arbitrary non-periodic function $P_\mu(S)$.

In the same way, the $\xi = 1$ equations are equivalent to the symmetry transformations with

$$\hat{H}'(\hat{X})_{\mu\lambda} = \delta_{\mu\lambda} + \left\{ |0\rangle \langle 0|, [\hat{X}_\lambda, \hat{X}_\nu] \right\} [\hat{X}_\mu, \hat{X}_\nu], \quad (102)$$

The difference of these transformations is the commutator

$$\hat{H}(\hat{X})_{\mu\lambda} - \hat{H}'(\hat{X})_{\mu\lambda} = \left[[\hat{X}_\mu, \hat{X}_\nu], \left\{ |0\rangle \langle 0|, [\hat{X}_\lambda, \hat{X}_\nu] \right\} \right]. \quad (103)$$

What is the meaning of these observations? The loop equations were derived as Schwinger-Dyson equations for the large N_c Yang-Mills theory. Apparently, this theory is equivalent to the Quantum Mechanics of the operators \hat{X} , with some hidden nonlinear symmetry. This is not a symmetry of the loop equations, this symmetry is *all* these equations are about.

Any representation of this symmetry solves the loop equations. We could introduce the generators $\hat{\Gamma}_\mu, \hat{\Gamma}'_\mu$ such that

$$\begin{aligned} [\hat{X}_\mu, \hat{\Gamma}_\lambda] &= \hat{H}(\hat{X})_{\mu\lambda}, \\ [\hat{X}_\mu, \hat{\Gamma}'_\lambda] &= \hat{H}'(\hat{X})_{\mu\lambda}, \\ \hat{\Gamma}_\lambda |0\rangle &= \hat{\Gamma}'_\lambda |0\rangle = 0, \\ \langle 0 | \hat{\Gamma}_\lambda &= \langle 0 | \hat{\Gamma}'_\lambda = 0. \end{aligned} \tag{104}$$

The Fock space is defined as the set of words

$$\begin{aligned} |\mu_1 \dots \mu_m\rangle &= a_{\mu_1}^\dagger \dots a_{\mu_m}^\dagger |0\rangle, \\ \langle \nu_n \dots \nu_1| &= \langle 0 | a_{\nu_n} \dots a_{\nu_1}, \\ \langle \nu_n \dots \nu_1 | \mu_1 \dots \mu_m \rangle &= \delta_{mn} \prod_i^n \delta_{\mu_i \nu_i}. \end{aligned} \tag{105}$$

The position operator according to Voiculescu can be chosen to be

$$\hat{X}_\mu = a_\mu + \sum_{k=1}^{\infty} Q_{\mu, \mu_1 \dots \mu_k} a_{\mu_1}^\dagger \dots a_{\mu_k}^\dagger, \tag{106}$$

where the first term reduces the word by one letter, and the second term expands it by arbitrary number of letters. That is

$$\begin{aligned} \langle \nu_n \dots \nu_1 | \hat{X}_\mu | \mu \nu_1 \dots \nu_n \rangle &= 1, \\ \langle \nu_n \dots \nu_1 | \hat{X}_\mu | \nu_k \dots \nu_n \rangle &= Q_{\mu, \nu_1 \dots \nu_{k-1}}, \end{aligned} \tag{107}$$

the rest of matrix elements vanishing. As for the generators, those have arbitrary matrix elements, between any finite words.

Let us write down explicit set of relations between matrix elements. First, consider the VEV of these equations, we get

$$\langle 0 | [\hat{X}_\mu, \hat{X}_\nu] [\hat{X}_\nu, \hat{X}_\lambda] | 0 \rangle = \delta_{\mu\lambda}. \tag{108}$$

We used the fact that $\langle 0 | [\hat{X}_\mu, \hat{X}_\nu] | 0 \rangle = 0$, which follows from space symmetry in more than two dimensions (in two dimensions this could be proportional to $\epsilon_{\mu\nu}$).

The next equation follows when the matrix elements between vacuum and arbitrary word are taken. We get

$$\langle 0 | \hat{X}_\mu \hat{\Gamma}_\lambda | \mu_1 \dots \mu_n \rangle = 0. \tag{109}$$

All the creation terms in \hat{X}_μ drop here, so we simply find

$$\langle \mu | \hat{\Gamma}_\lambda = 0. \tag{110}$$

In other words, not only vacuum state is annihilated by the generator $\hat{\Gamma}_\lambda$, any (left) letter is annihilated as well. For the second generator $\hat{\Gamma}'_\lambda$ we get the similar equation

$$\hat{\Gamma}'_\lambda \hat{X}_\mu |0\rangle = 0. \quad (111)$$

However, the operator \hat{X}_μ generates the tower of word states from the right vacuum, so that this equation is not so simple.

Finally, the non-vacuum matrix elements of the symmetry relations read

$$\begin{aligned} \langle W | [\hat{X}_\mu, \hat{\Gamma}_\lambda] | W' \rangle &= \delta_{\mu\lambda} \delta_{W,W'} + \langle W | [\hat{X}_\mu, \hat{X}_\nu] | 0 \rangle \langle [\lambda, \nu] | W' \rangle, \\ \langle W | [\hat{X}_\mu, \hat{\Gamma}'_\lambda] | W' \rangle &= \delta_{\mu\lambda} \delta_{W,W'} + \langle W | [\hat{X}_\lambda, \hat{X}_\nu] | 0 \rangle \langle [\mu, \nu] | W' \rangle, \end{aligned} \quad (112)$$

with obvious notation

$$\langle [\mu, \nu] \rangle \equiv \langle \mu\nu \rangle - \langle \nu\mu \rangle. \quad (113)$$

Note that these equations are only quadratic in unknown matrix elements of \hat{X} . All the higher order terms canceled. Note also that there is an infinite set of bilinear equations

$$\langle W | [\hat{X}_\mu, \hat{\Gamma}_\lambda] | W' \rangle = \langle W | [\hat{X}_\mu, \hat{\Gamma}'_\lambda] | W' \rangle = \delta_{\mu\lambda} \delta_{W,W'}, \quad (114)$$

for all the words W' with $n \neq 2$ letters. The truly nonlinear equations are those with $n = 2$

$$\begin{aligned} \langle W | [\hat{X}_\mu, \hat{\Gamma}_\lambda] | \alpha\beta \rangle &= \delta_{\mu\lambda} \langle W | \alpha\beta \rangle + \langle W | [\hat{X}_\mu, \hat{X}_\beta] \delta_{\lambda\alpha} - [\hat{X}_\mu, \hat{X}_\alpha] \delta_{\lambda\beta} | 0 \rangle, \\ \langle W | [\hat{X}_\mu, \hat{\Gamma}'_\lambda] | \alpha\beta \rangle &= \delta_{\mu\lambda} \langle W | \alpha\beta \rangle + \langle W | [\hat{X}_\lambda, \hat{X}_\beta] \delta_{\mu\alpha} - [\hat{X}_\lambda, \hat{X}_\alpha] \delta_{\mu\beta} | 0 \rangle. \end{aligned} \quad (115)$$

At present we do not know how to solve these equations in closed form. However, the recurrent equations for the correlation functions of products of \hat{X} can be derived and solved one after another.

11 Operator Expansion

These recurrent equations follow directly from the symmetry property

$$\sum_{l=1}^n \langle 0 | \hat{X}_{\mu_1} \dots \hat{H}(\hat{X})_{\mu_l \lambda} \dots \hat{X}_{\mu_n} | 0 \rangle = 0. \quad (116)$$

At $n = 1$ we have the previous equation (108). In the higher orders we get similar recurrent equations. The highest rank tensor here come from the term with $l = n$. Moving all the rest to the right-hand side and skipping the terms which vanish identically, and using the $n = 1$ equation we get a recurrent equation

$$\langle 0 | 0 \rangle \langle 0 | \hat{X}_{\mu_1} \dots \hat{X}_{\mu_{n-1}} [\hat{X}_{\mu_n}, \hat{X}_\nu] [\hat{X}_\lambda, \hat{X}_\nu] | 0 \rangle = \quad (117)$$

$$\begin{aligned}
& - \sum_{l=2}^n \delta_{\mu_l \lambda} \langle 0 | \hat{X}_{\mu_1} \dots \hat{X}_{\mu_{l-1}} \hat{X}_{\mu_{l+1}} \dots \hat{X}_{\mu_n} | 0 \rangle \\
& - \sum_{l=3}^{n-2} \langle 0 | \hat{X}_{\mu_1} \dots [\hat{X}_{\mu_l}, \hat{X}_\nu] | 0 \rangle \langle 0 | [\hat{X}_\lambda, \hat{X}_\nu] \dots \hat{X}_{\mu_n} | 0 \rangle \\
& - \sum_{l=3}^{n-2} \langle 0 | \hat{X}_{\mu_1} \dots [\hat{X}_{\mu_l}, \hat{X}_\nu] [\hat{X}_\lambda, \hat{X}_\nu] | 0 \rangle \langle 0 | \dots \hat{X}_{\mu_n} | 0 \rangle.
\end{aligned}$$

The second transformation leads to the double commutator equation

$$\sum_{l=1}^n \langle 0 | \hat{X}_{\mu_1} \dots [[\hat{X}_{\mu_l}, \hat{X}_\nu], \{ |0\rangle \langle 0|, [\hat{X}_\lambda, \hat{X}_\nu] \}] \dots \hat{X}_{\mu_n} | 0 \rangle = 0. \quad (118)$$

There are two highest rank tensor terms, which cancel in virtue of cyclic symmetry. Therefore, the second equation reads

$$\sum_{l=3}^{n-2} \langle 0 | \hat{X}_{\mu_1} \dots [[\hat{X}_{\mu_l}, \hat{X}_\nu], \{ |0\rangle \langle 0|, [\hat{X}_\lambda, \hat{X}_\nu] \}] \dots \hat{X}_{\mu_n} | 0 \rangle = 0 \quad (119)$$

The first terms of this expansion are easy to find. The most general solution for the four point function reads

$$\langle 0 | \hat{X}_{\mu_1} \hat{X}_{\mu_2} \hat{X}_{\mu_3} \hat{X}_{\mu_4} | 0 \rangle = -\frac{1}{2(d-1)} \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} + \text{symmetric}. \quad (120)$$

The symmetric terms, proportional to

$$\delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} + \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} + \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3}, \quad (121)$$

are left ambiguous. These are the gauge terms, which cancel in observables by virtue of translational symmetry (93). The correlation functions of Abelian operators $\langle C_{\mu_1} \dots C_{\mu_n} \rangle$ produce symmetric terms.

In four dimensions there is an extra term

$$N_c g_0^2 \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \quad (122)$$

in the four point function, coming from chiral anomaly. In two dimension the similar term

$$N_c g_0^2 \epsilon_{\mu_1 \mu_2} \quad (123)$$

would contribute to the two point function.

The anomalous terms have perturbative normalizations, not affected by effects of strong interactions. The normalization can be recovered from (53), keeping in mind the extra factor of $\frac{1}{N_c g_0^2}$ in definition of the vacuum average (91).

The two-point correlation in four dimensions remains undetermined. This introduces arbitrary parameter of dimension of square of length

$$\langle 0 | \hat{X}_\mu \hat{X}_\nu | 0 \rangle = A \langle 0 | 0 \rangle \delta_{\mu\nu}. \quad (124)$$

In the $n = 3$ order we get the equation for the 6-th rank tensor

$$\left\langle 0 \left| \hat{X}_{\mu_1} \hat{X}_{\mu_2} \left[\hat{X}_{\mu_3}, \hat{X}_{\nu} \right] \left[\hat{X}_{\lambda}, \hat{X}_{\nu} \right] \right| 0 \right\rangle = -A (\delta_{\mu_1 \mu_2} \delta_{\mu_3 \lambda} + \delta_{\mu_1 \mu_3} \delta_{\mu_2 \lambda}). \quad (125)$$

The computations of these tensors was done by means of the *Mathematica* algorithms of the previous work (see files attached to the hep-th source of [1]). Again, there are some free coefficients, which (partly) drop in observables.

The non-commutative probability theory goes as usual with this definition of \hat{X} correlators. The operator expansion goes as above, with coefficients Q_{μ_n, \dots, μ_1} given by planar connected moments $\left\langle 0 \left| \hat{X}_{\mu_1} \dots \hat{X}_{\mu_n} \right| 0 \right\rangle_{pl.conn.}$.

12 Glueball Equation

As it was pointed out in [1] the equation for the glueball excitation wave-function $\Psi[C]$ corresponds to linearization of the loop equation, which can be achieved by variations of the position operator. Let us elaborate this idea in the context of the supermomentum loop equation.

First, let us identify the total momentum dependence of the glueball loop equation. The coordinate superloop equation had the form of

$$\hat{L}_C(\mathcal{W}[\cdot]) = \hat{R}_C(\mathcal{W}[\cdot], \mathcal{W}[\cdot]), \quad (126)$$

where \hat{L} is a linear operator, and \hat{R} is a bilinear one.

The initial form of the \hat{L} operator was

$$\hat{L}_C(\mathcal{W}[\cdot]) = \int_{-0}^{+0} dS \frac{\delta \mathcal{W}[X]_0^1}{\delta X_\mu(0) \delta X_\mu(S)}, \quad (127)$$

which involved integration over infinitesimal interval $(-0, +0)$ around the point 0 where the first variation took place. This operator is local, it picks only the delta terms, related to the Y-M equations of motion.

Now, in virtue of translation invariance

$$\oint dS \frac{\delta \mathcal{W}[C]}{\delta X_\mu(S)} = 0, \quad (128)$$

which allowed us to replace the local operator by a nonlocal one

$$\hat{L}_C(\mathcal{W}[\cdot]) = - \int_0^1 dS \frac{\delta \mathcal{W}[X]_0^1}{\delta X_\mu(0) \delta X_\mu(S)}, \quad (129)$$

with integration covering the rest of the loop. This was the operator we used above, in transforming superloop equation to the momentum space.

When the glueball wave function is studied, the translation invariance is replaced by fixed momentum condition

$$\oint dS \frac{\delta \Psi[C]}{\delta X_\mu(S)} = i k_\mu \Psi[C]. \quad (130)$$

This will result in extra term in the momentum loop operator

$$\hat{L}_P(\Psi) = DP_\mu(0) \left(-k_\mu + \int_0^1 dP_\mu(S) \right) \Psi[P]. \quad (131)$$

Repeating all the previous steps for the linearized momentum loop equation, we find

$$k_\lambda \Psi[P]_0^1 = \int_0^1 dP_\mu(S) \left(\delta_{\lambda\mu} \Psi[P]_0^1 + N_c g_0^2 \frac{\partial \left(\mathcal{C}[P]_0^S \Psi[P]_S^1 + \Psi[P]_0^S \mathcal{C}[P]_S^1 \right)}{\partial \Sigma_{\mu\nu}(\xi) \partial \Sigma_{\lambda\nu}(S)} \right). \quad (132)$$

Let us prove that the following Ansatz solves this equation

$$\Psi[P]_A^B = \int_A^B dP_\mu(S) \left\langle 0 \left| \hat{T} \exp \left(\imath \int_A^S \hat{X}_\nu dP_\nu(V) \right) \hat{Y}_\mu \hat{T} \exp \left(\imath \int_S^B \hat{X}_\lambda dP_\lambda(W) \right) \right| 0 \right\rangle, \quad (133)$$

provided the operator \hat{Y}_λ satisfies the commutation relations

$$[\hat{\Gamma}_\lambda, \hat{Y}_\mu] = [\hat{\Gamma}'_\lambda, \hat{Y}_\mu] = k_\lambda \hat{Y}_\mu. \quad (134)$$

The idea is to take a variation $\delta \mathcal{C}[P] = \Psi[P]$ of initial operator representation (91) with $\delta \hat{X}_\mu = N_c g_0^2 \hat{Y}_\mu$, to reproduce the right side of the glueball loop equation (132). The left side comes about from the commutator (134).

The formal proof follows from the identity

$$0 \equiv \int_0^1 dP_\mu(S) \left\langle 0 \left| \left[\hat{T} \exp \left(\imath \int_0^S \hat{X}_\nu dP_\nu(V) \right) \hat{Y}_\mu \hat{T} \exp \left(\imath \int_S^1 \hat{X}_\lambda dP_\lambda(W) \right), \hat{\Gamma}_\lambda \right] \right| 0 \right\rangle \quad (135)$$

and the same for $\hat{\Gamma}'_\lambda$. Commuting the generators with ordered exponentials as follows

$$\begin{aligned} & \left[\hat{T} \exp \left(\imath \int_A^B \hat{X}_\nu dP_\nu(V) \right), \hat{\Gamma}_\lambda \right] = \\ & \imath \int_A^S dP_\mu(U) \hat{T} \exp \left(\imath \int_A^U \hat{X}_\alpha dP_\alpha(W) \right) \hat{H}_{\mu\lambda}(\hat{X}) \hat{T} \exp \left(\imath \int_U^B \hat{X}_\beta dP_\beta(T) \right). \end{aligned} \quad (136)$$

and using the definitions (98),(102), (134) and algebra of differential forms (25),(27), it is straightforward to verify the glueball loop equation.

The linear equations (134) represent the eigenvalue equations for the glueball spectrum in this theory. The states $\hat{Y}_\mu |0\rangle$ represent the momentum eigenstates,

$$\begin{aligned} \hat{\Gamma}_\alpha \hat{Y}_\mu |0\rangle &= k_\alpha \hat{Y}_\mu |0\rangle, \\ \hat{\Gamma}'_\alpha \hat{Y}_\mu |0\rangle &= k_\alpha \hat{Y}_\mu |0\rangle. \end{aligned} \quad (137)$$

We see, that operators $\hat{\Gamma}_\alpha, \hat{\Gamma}'_\alpha$ play the role of spatial translation generators of the endpoint of “QCD string”. They have to be found from above nonlinear operator algebra. In general, they do not commute, so these are not the Poincare translation generators.

Clearly, the commutators must annihilate these states

$$\begin{aligned} [\hat{\Gamma}_\alpha, \hat{\Gamma}_\beta] \hat{Y}_\mu |0\rangle &= 0, \\ [\hat{\Gamma}'_\alpha, \hat{\Gamma}'_\beta] \hat{Y}_\mu |0\rangle &= 0, \\ [\hat{\Gamma}_\alpha, \hat{\Gamma}'_\beta] \hat{Y}_\mu |0\rangle &= 0. \end{aligned} \tag{138}$$

which yields restrictions on the Voiculescu coefficients of \hat{Y}_μ operators.

13 Quark Propagating Around the Superloop

Let us now have a look at the negative parity current correlators, which up to normalization and standard Chan-Paton factor $\text{tr } \tau_{a_1} \dots \tau_{a_n}$ are given by the following integral

$$\Gamma_n^{\nu_1 \dots \nu_n}(k_1, \dots k_n) = \int_0^1 \frac{\partial}{\partial K_{\nu_1}(S_1)} \dots \frac{\partial}{\partial K_{\nu_n}(S_n)} \left\langle 0 \left| \hat{T} \exp \left(\int_0^1 dK_\mu(S) \hat{X}_\mu \right) \right| 0 \right\rangle, \tag{139}$$

where it is implied that after taking all functional derivatives the supermomentum

$$K_\mu(S) = \sum_{i=1}^n k_\mu^i \Theta(S_i, S). \tag{140}$$

So, the value of the supermomentum also depends upon $S_1, \dots S_n$.

There is a systematic way of handling such superpath the ordered exponentials, suggested in [11]. In our context this looks as follows. We introduce the Quark superfield

$$Q_W(S) = q_W(s) + \theta \zeta_W(s) \tag{141}$$

which is a $|ket\rangle$ vector in the Fock space of words $W = |\mu_1 \dots \mu_n\rangle$, and respectively

$$\bar{Q}_W(S) = \bar{q}_W(s) + \theta \bar{\zeta}_W(s), \tag{142}$$

which is a $\langle bra|$ vector. Then we represent the SMLoop as follows

$$\begin{aligned} \left\langle 0 \left| \hat{T} \exp \left(\int_0^1 dK_\mu(U) U \hat{X}_\mu \right) \right| 0 \right\rangle &= \\ \int \mathcal{D}\bar{Q} \mathcal{D}Q \exp \left(\int_0^1 \bar{Q}(U) \left(dQ(U) - \int_0^1 dK_\mu(U) \hat{X}_\mu Q(U) \right) \right) &\langle 0 | Q(0) \otimes \bar{Q}(1) | 0 \rangle. \end{aligned} \tag{143}$$

The proof essentially repeats the arguments of [11], with some improvements and corrections. For brevity we will skip the indexes W , so that, e.g. the propagator

$$G(S, S') = \langle Q(S) \otimes \bar{Q}(S') \rangle_{\bar{Q}, Q}, \tag{144}$$

represents the operator in Fock space with matrix elements

$$\langle W | G(S, S') | W' \rangle = \langle \mu_1 \dots \mu_n | G(S, S') | \mu'_1 \dots \mu'_{n'} \rangle. \tag{145}$$

The bare propagator $G^0(S, S')$ of this Quark satisfies an equation

$$DG^0(S, S') = \delta(S' - S), \quad (146)$$

which we know how to solve

$$G^0(S, S') = \Theta(S, S'). \quad (147)$$

The higher vacuum loops all vanish by virtue of the identities

$$\Theta(S, S')\Theta(S', S) = 0, \quad (148)$$

$$\Theta(S, S')\Theta(S', S'')\Theta(S'', S) = 0,$$

...

This leaves only one vacuum loop, corresponding to first order diagram

$$\int_0^1 dK_\mu(S)\Theta(S, S)\text{tr } X_\mu = \frac{1}{2} \int_0^1 dK_\mu(S)\text{tr } X_\mu \quad (149)$$

which vanishes by virtue of periodicity of $K_\mu(S)$, (plus the trace vanishes due to Lorentz symmetry).

The complete Greens function is a solution of the equation

$$DG(S, S') = \delta(S' - S) - \imath (DK_\mu(S))\hat{X}_\mu G(S, S'). \quad (150)$$

The formal solution of this equation is given by the the ordered exponential

$$G(S, S') = \hat{T} \exp \left(\imath \int_S^{S'} dK_\mu(U)\hat{X}_\mu \right), \quad (151)$$

which is just what we need.

The variations with respect to supermomentum now act on usual rather than the ordered exponential with the result

$$\imath \frac{\partial}{\partial K_\mu(S)} \Rightarrow d(\bar{Q}(S)X_\mu Q(S)) = d\bar{Q}(S)X_\mu Q(S) + \bar{Q}(S)X_\mu dQ(S), \quad (152)$$

which plays the role of the vector current of our superloop theory.

The multicurrent amplitude reads

$$\Gamma_n^{\nu_1 \dots \nu_n}(k_1, \dots k_n) = \int_0^1 \left\langle d^n \left(\bar{Q}(S)X_{\{\nu\}} Q(S) \right) \langle 0 | Q(0) \otimes \bar{Q}(1) | 0 \rangle \right\rangle_{\bar{Q}, Q}. \quad (153)$$

One may try to integrate by parts here, replacing the covariant derivatives of Quark currents by those acting on the Θ functions in our the ordered integral. However, there is some more S_i dependence, which comes from external supermomentum $K_\mu(S)$. This brings down more Quark currents $D\bar{Q}\hat{X}_\mu Q$, which does not simplify formulas.

The Gaussian average over \bar{Q}, Q reduces to super-graphs with propagator $G(S, S')$ in external field $\hat{X}_\mu DK_\mu(S)$. Now, once all variations are done, we can set the supermomentum to its value, after which this external field reduces to the sum of δ functions in superspace.

After this the ordered exponential of the integral of sum of the δ function terms will reduce to the product of the vertex operators $\prod_l \exp \left(i k_\mu^l \bar{Q}(S_l) \hat{X}_\mu Q(S_l) \right)$.

The flavor vector current correlation functions are given by the multiple superspace integrals of the \bar{Q}, Q average. This average reduces to sum of the *connected graphs*. All the vacuum graphs vanish in our 1D supergravity, as discussed above.

The generic connected graph looks the same as the string theory graphs, with currents represented by

$$J_\mu^a(k, S) \rightarrow \tau^a \otimes d(\bar{Q}(S) X_\mu Q(S)) \exp \left(i k_\mu \bar{Q}(S) \hat{X}_\mu Q(S) \right), \quad (154)$$

and propagators $\Theta(S, S')$.

14 SUSY and Chiral Symmetry

In the vector sector (anti-periodic boundary conditions) we have to keep the kinetic term in the loop Lagrangian.

The general formula (47) for the current correlations involves an extra operator Z , which we replaced by 1 in the topological sector. Using the above quark superfield theory we could replace this operator by

$$\begin{aligned} Z &= \lim_{\alpha \rightarrow 0} \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) \exp \left(-T m_0^2 - \frac{\hat{L}}{T} \right), \\ \hat{L} &= \frac{1}{4} \int_0^1 dS D \left(\bar{Q}(S) \hat{X}_\mu Q(S) \right) D^2 \left(\bar{Q}(S) \hat{X}_\mu Q(S) \right), \end{aligned} \quad (155)$$

which will result in extra quartic interaction.

What could be an effect of this interaction? Formally, one can develop the perturbation expansion and compute all the arising T integrals

$$\int_0^\infty dT T^{\alpha-n-1} (1 + \alpha \ln T) \exp \left(-T m_0^2 \right) \rightarrow \frac{(-m_0^2)^n}{n!} \left(\frac{\psi(n+1)}{n!} - \ln m_0^2 \right). \quad (156)$$

We see, that these integrals contain the increasing powers of the bare mass.

However, the remaining superspace integrals may diverge, because the local SUSY is broken down to the global one. Specifically, the singularities at small intervals $s - s' \rightarrow 0$ now do not match the number of integrations. Extra operator $D^2 = \frac{\partial}{\partial s}$ produces ultraviolet divergences. Part of these divergences is for real— one has to renormalize the mass after all.

The spontaneous chiral symmetry breaking must come about as a singularity $O\left(\sqrt{m_0^2}\right)$ in normal channels, but also as extra poles at zero momenta in the pion channels of vector current amplitudes we consider.

This means that the self energy, induced by this quartic interaction, can remain finite in the zero mass limit, and move the pole to zero momentum. By power counting it is quite plausible. The bubble diagrams will contain the singularities, which presumably will be cut at the new scale, induced by the mass. Solving the Dyson equation for this new mass scale may lead to the chiral symmetry breaking in the limit $m_0 \rightarrow 0$.

The divergences produced by the physical mass term are unpleasant. On the other hand we know that the vector sector in the chiral limit is finite. The pion poles involve the finite residues which do not need extra renormalization. Is there a way to avoid these spurious divergences?

We suggest the following soft chiral symmetry breaking, without unnecessary violations of local SUSY. Let us add the constant Abelian external field with finite density of topological charge. The chiral symmetry will be broken by the Adler-Bell-Jackiw anomaly, so that there will be no massless pion.

The vacuum energy can be written as the sum of the zero mode contribution and the finite modes. In the limit of the vanishing quark mass

$$\begin{aligned}\mathcal{E} &= -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda (\nu_0 \delta(\lambda) + \rho(\lambda)) \ln(m_0^2 + \lambda^2) \\ &= -\nu_0 \ln m_0 - \frac{1}{2} \int_0^{\infty} d\lambda \rho(\lambda) \ln(m_0^2 + \lambda^2) \rightarrow -\nu_0 \ln m_0 + \text{const},\end{aligned}\tag{157}$$

where $\nu_0 > 0$ is the net number of zero modes. The leading divergent term in the chiral limit is the contribution of the zero modes. Unlike the chiral anomaly, the net number of zero modes is not a topological invariant.

Let us study this number in the free quark theory. The corresponding Euler-Heisenberg Lagrangian was computed in Appendix B. The formula (170) can be written as

$$V \int_0^{\infty} dT T^{\alpha-1} (1 + \alpha \ln T) e^{-Tm_0^2} \prod_{i=1}^{\frac{1}{2}d} b_i \coth(2b_i T),\tag{158}$$

where $\pm i b_i$ are the eigenvalues of $B_{\mu\nu}$. In the chiral limit the leading logarithmic term comes from the region $b_i^{-1} \ll T \ll m_0^{-2}$ where one could replace $\coth(2b_i T) \rightarrow \text{sign}(b_i)$, after which the integral yields

$$V \prod |b_i| \int_0^{\infty} dT T^{\alpha-1} (1 + \alpha \ln T) e^{-Tm_0^2} \rightarrow -(\gamma + \ln m_0^2) V \left| \sqrt{\det i B} \right|.\tag{159}$$

This is an absolute value of the Pfaffian.

The same result (up to normalization) could be obtained much simpler, by neglecting the kinetic energy in the path integral. The path integral becomes supersymmetric,

$$\begin{aligned}V \int \mathcal{D}X \delta^d(X(0)) \exp \left(\frac{i}{2} B_{\mu\nu} \int_0^1 dX_\mu(S) X_\nu(S) \right) &= \\ V \int \mathcal{D}X \delta^d(X(0)) \exp \left(\sum_{i=1}^{\frac{1}{2}d} b_i \int_0^1 dX_{2i-1}(S) X_{2i}(S) \right) &\propto V \prod_{i=1}^{\frac{1}{2}d} |b_i| = V \left| \sqrt{\det i B} \right|.\end{aligned}\tag{160}$$

We used the rotational invariance to reduce the antisymmetric quadratic form to the Jordan form. Then we rescaled half of components of the superfield $X_{2i}(S) \rightarrow b_i^{-1} X_{2i}(S)$. The SUSY measure stays invariant, because $STr 1 = 0$, therefore the only contribution comes from rescaling of the δ function. The proper time integral gives then the logarithmic contribution.

So, we get the absolute value of the topological charge density. The vacuum average of this number in QCD does not vanish, unlike the average topological charge. In the background of external constant flavor field with nontrivial topology we expect this number to be proportional to the volume of space, which makes it a leading term in the chiral limit (it is proportional to $\ln m_0^2$, unlike the contribution from the finite modes).

We could argue that the quark kinetic energy drops in QCD by the same reason as in the Euler-Heisenberg Lagrangian, and we arrive at the locally SUSY theory in the vector sector

$$\Gamma_n^{\nu_1 \dots \nu_n}(k_1, \dots k_n) = -\ln m_0 \int_0^1 \left\langle \langle 0 | Q(0) \otimes \bar{Q}(1) | 0 \rangle \prod_{i=1}^n d(\bar{Q}(S_i) X_{\nu_i} Q(S_i)) \right\rangle_{\bar{Q}, Q}. \quad (161)$$

with the usual supermomentum

$$K_\mu(U) = \sum_{i=1}^n k_\mu^i \Theta(S_i, U). \quad (162)$$

The corresponding term in the Quark action also represents the quartic interaction

$$\frac{i}{2} B_{\mu\nu} \int dS D \left(\bar{Q}(S) \hat{X}_\nu Q(S) \right) \left(\bar{Q}(S) \hat{X}_\mu Q(S) \right), \quad (163)$$

which preserves the local SUSY. The only symmetry which gets broken is the Lorentz symmetry, as we now have external tensor field.

One could study the perturbation expansion in this quartic interaction and try to identify and sum up the terms responsible for the spontaneous chiral symmetry breaking in the limit $B_{\mu\nu} \rightarrow 0$. The good thing about this perturbation expansion is that all the graphs will still be locally SUSY, and therefore calculable.

This interesting issue calls for a further study.

15 Summary and Conclusions

Let us summarize the status of the superloop theory of the large N_c QCD. Here are the basic features of this theory.

- The chiral momentum loop amplitudes, with projector $\frac{1}{2}(1 \pm \gamma_5)$ inside the Dirac trace, satisfy the closed equation, generalizing the ordinary momentum loop equation, for the case of the supermomentum loops.

- The supermomentum loop equation can be interpreted as a nonlinear symmetry relations (104) of the position operator \hat{X}_μ of the QCD string. This symmetry lead to recurrent relations for the coefficients of the Taylor expansion of momentum superloop.
- The (non-commutative) generators $\hat{\Gamma}_\mu, \hat{\Gamma}'_\mu$ of these symmetry transformations can be interpreted as string endpoint translation operators. In particular, the glueball excitations with fixed momentum k_μ are related to the eigenstates of these operators with common vector eigenvalue k_μ .
- The position operator can be represented as a_μ plus expansion in a_μ^\dagger . These operators satisfy Cuntz algebra and annihilate the vacuum $|0\rangle$. Generic state is a word $\prod a_{\mu_i}^\dagger |0\rangle$. Expansion coefficients are given by planar connected moments of $\langle 0 | \prod \hat{X}_{\mu_i} | 0 \rangle$.
- The correlators of the vector quark currents $\langle \prod J_{\mu_i}^a(k^i) \rangle$ in the chiral limit in the background of strong constant electro-magnetic field are related to (functional derivatives of) the SMLoop with supermomentum $K_\mu(S) = \sum_i k_\mu^i \Theta(S_i, S)$.
- One could introduce the Quark superfield at the loop $Q(S)$ in such a way that the flavor vector current is represented at the loop as (154).
- In the chiral limit the Quark superfield is Gaussian, with the propagator, depending on \hat{X}_μ and momenta of all the currents. The exact operator solution for this propagator looks very similar to the good old string diagrams.
- The low-energy expansion of flavor current correlators in the chiral limit in the background of strong constant electro-magnetic field is also calculable in terms of these coefficients. There are no singular integrals involved. All the superloop integrals reduce to rational numbers. These tedious calculations can be simplified by means of generating function derived in Appendix E.
- In order to account for the chiral symmetry breaking effects in anomalous channels of these current correlators, one should add the quark mass m_0 . This results in extra quartic interaction in our 1D superfield theory. The interaction constant is proportional to the m_0^2 . This interaction breaks the local SUSY down to the global one, and introduces the extra divergences. Presumably, summing the leading divergences (bubble graphs?) will account for the chiral symmetry breaking.

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A One More Representation of a Log

Let us consider the identity

$$A^{-\alpha}\Gamma(\alpha) = \int_0^\infty dT T^{\alpha-1} e^{-AT}, \quad (164)$$

valid for any positive definite operator A . Multiplying it by α and taking derivative $\frac{\partial}{\partial \alpha}$ we find

$$A^{-\alpha}\Gamma(1+\alpha)(\psi(1+\alpha) - \ln A) = \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) e^{-AT}. \quad (165)$$

Tending $\alpha \rightarrow +0$,

$$\gamma + \ln A = - \lim_{\alpha \rightarrow +0} \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) e^{-AT}. \quad (166)$$

The formula in the text corresponds to $A = m_0^2 + (i\gamma_\mu \nabla_\mu)^2$ which is positive definite regardless of the possible zero modes of the Dirac operator.

B Euler-Heisenberg Lagrangian

Let us compute the vacuum energy in constant Abelian field $B_\nu(X) = \frac{i}{2} B_{\mu\nu} X_\mu$.

The quadratic forms in exponential of (37) are the same for Bose and Fermi parts

$$\begin{aligned} \int_0^1 ds \left(x'_\mu(s) Q_{\mu\nu} \left(\frac{\partial}{\partial s} \right) x_\nu(s) + \psi'_\mu(s) Q_{\mu\nu} \left(\frac{\partial}{\partial s} \right) \psi_\nu(s) \right), \\ Q_{\mu\nu}(\omega) = \frac{\omega}{4T} \delta_{\mu\nu} + \frac{i}{2} B_{\mu\nu}, \end{aligned} \quad (167)$$

with the only difference coming from the boundary conditions.²

Let us expand the fields in Fourier expansion on the circle

$$\begin{aligned} x_\mu(s) &= \sum_{n=-\infty}^{\infty} x_\mu^n \exp(2n\pi s), \\ \psi_\mu(s) &= \sum_{n=-\infty}^{\infty} \psi_\mu^n \exp((2n+1)\pi s), \end{aligned} \quad (168)$$

then the functional integral yields the product of ratio of the determinants for each Fourier mode

$$V \sqrt{\det Q(0)} \prod_{n=-\infty}^{\infty} \frac{\sqrt{\det Q((2n+1)\pi)}}{\sqrt{\det Q(2n\pi)}}. \quad (169)$$

The first factor corrects the contribution from the zero mode x_μ^0 . There is, in fact, no Gaussian integration for this mode in our action, so that this integration produces the volume V of space. The factor $\sqrt{\det Q(0)}$ removes the zero mode factor from the denominator at $n = 0$ in the product.

²Let us note in passing that the periodic boundary conditions $\psi(1) = \psi(0)$ correspond to the chiral anomaly instead of the vacuum energy. We discuss this issue later in some detail.

Now, the standard computation of the infinite product yields

$$V \int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) e^{-T m_0^2} \sqrt{\det(B \cot 2BT)}. \quad (170)$$

which is our representation of the Euler-Heisenberg Lagrangian. The determinant of the matrix function is implied here. One could expand it in power series in B (we denote the Bernoulli numbers as \mathcal{B}_{2n})

$$B \cot 2BT = \frac{1}{2T} \left(1 - \sum_{n=1}^{\infty} \frac{|\mathcal{B}_{2n}|}{(2n)!} (4BT)^{2n} \right), \quad (171)$$

$$\sqrt{\det(B \cot 2BT)} = T^{-\frac{1}{2}d} \sum_{n=0}^{\infty} C_{2n}(B) T^{2n},$$

with some invariant polynomials

$$C_{2n}(B) = \sum_{\{k\}} c_{\{k\}} \prod \left(\text{tr } B^{2i} \right)^{k_i}. \quad (172)$$

The proper time integrals

$$\int_0^\infty dT T^{\alpha-1} (1 + \alpha \ln T) e^{-T m_0^2} T^{2n-\frac{1}{2}d} = \frac{\partial}{\partial \alpha} \left(\alpha m_0^{d-4n-2\alpha} \Gamma \left(2n + \alpha - \frac{1}{2}d \right) \right). \quad (173)$$

The pole at $d = 4$ in the B^2 term is the usual UV divergence, removed by our α renormalization as follows

$$\frac{\partial}{\partial \alpha} \left(\alpha m_0^{-2\alpha} \Gamma(\alpha) \right) \rightarrow -\gamma - 2 \ln m_0. \quad (174)$$

C Supersymmetrization Theorem

Let us consider the function $F(S_1, \dots, S_n)$ of n variables $S_i = (s_i, \theta_i)$ which linearly transforms under infinitesimal local SUSY transformations

$$\begin{aligned} \delta s &= \epsilon \theta \beta(s), \\ \delta \theta &= \epsilon \beta(s). \end{aligned} \quad (175)$$

The finite transformation have the form

$$\begin{aligned} s(\lambda) &= s + \lambda \theta \beta(s), \\ \theta(\lambda) &= \theta + \lambda \beta(s) + \frac{1}{2} \lambda^2 \theta \beta(s) \beta'(s). \end{aligned} \quad (176)$$

One may verify that these polynomials satisfy differential equation

$$\begin{aligned} s'(\lambda) &= \theta(\lambda) \beta(s(\lambda)), \\ \theta'(\lambda) &= \beta(s(\lambda)). \end{aligned} \quad (177)$$

The local SUSY implies that the equation $F(S_1(\lambda), \dots, S_n(\lambda)) = 0$ is, in fact, independent of λ for arbitrary Grassmann function $\beta(s)$. Using this independence, we may choose $\beta(s)$ so that it "passes through all the θ points"

$$\beta(s_i) = \theta_i. \quad (178)$$

An explicit example of such function is a Lagrange interpolating polynomial

$$\beta(s) = \sum_i \theta_i \prod_{j \neq i} \frac{s - s_j}{s_i - s_j}. \quad (179)$$

Now, taking $\lambda = -1$ we observe that

$$\theta_i(-1) = 0, s_i(-1) = s_i. \quad (180)$$

So, the function, which vanishes identically at the real axis, will vanish identically in whole superspace. Therefore, one can extend in superspace any invariant identity which holds on a real axis.

Being valid at any finite n , in a limit it must be also valid for 1D functionals $F[\Gamma]$ which depend on some curve $\Gamma : S = (s(\tau), \theta(\tau))$ in superspace. The same transformation with $\lambda = -1$ moves this curve to the real axis provided

$$\beta(s(\tau)) = \theta(\tau). \quad (181)$$

Any solution of for inverse function $\tau(s)$ will give an example of such transformation

$$\beta(s) = \theta(\tau(s)). \quad (182)$$

Such an obvious theorem must, of course, be well-known to mathematicians, but for a lazy physicist it was easier to "prove" it than to dig into the mathematical literature.

D Factorization of Fourier Integral

The factorization of the Fourier exponentials

$$\exp \left(\imath \int_A^B dP_\mu(U) X_\mu(U) \right) = \exp \left(\imath \int_A^S dP_\mu(V) X_\mu(V) \right) \exp \left(\imath \int_S^B dP_\mu(W) X_\mu(W) \right), \quad (183)$$

follows from the additivity of the dX_μ form. When we compute the gradient precisely in a breaking point, in a complete integral it is done by means of the integration by parts, with the usual result

$$\exp \left(-\imath \int_A^B dP_\mu(U) X_\mu(U) \right) \frac{\partial}{\partial P_\mu(S)} \exp \left(\imath \int_A^B dP_\mu(U) X_\mu(U) \right) = -\imath dX_\mu(S). \quad (184)$$

In a split form one can obtain the same result in one plays by the rules. This computation goes as follows

$$\begin{aligned}
& -\imath \exp \left(-\imath \int_A^S dP_\mu(V) X_\mu(V) \right) \frac{\delta}{\delta P_\mu(S)} \exp \left(\imath \int_A^S dP_\mu(V) X_\mu(V) \right) = \\
& \int_A^S d\delta(V-S) X_\mu(V) = \delta(0) X_\mu(S) - \delta(A-S) X_\mu(A) - \int_A^S dX_\mu(V) \delta(V-S) = \\
& - \int_A^S dX_\mu(V) \delta(V-S) = - \int dV DX_\mu(V) \delta(V-S) \Theta(A, V) \Theta(V, B) = \\
& -DX_\mu(S) \Theta(A, S) \Theta(S, S) = -\frac{1}{2} DX_\mu(S) \Theta(A, S).
\end{aligned} \tag{185}$$

The second exponential gives another half of the derivative $DX_\mu(S)$.

The same result could be obtained much simpler, using the Fourier $\frac{1}{2}$ rule

$$DX_\mu(S) = \frac{1}{2} DX_\mu(S-) + \frac{1}{2} DX_\mu(S+). \tag{186}$$

This $\frac{1}{2}$ rule is a general rule to interpret the discontinuities.

The terms with $S-$ ($S+$) is the boundary value of the ordinary derivative, acting on the left (right) exponential, where they can be replaced by functional derivatives $\frac{\delta}{\delta P_\mu(S\pm)}$ for the inside points. The left derivative $\frac{\delta}{\delta P_\mu(0-)}$ is equivalent to $\frac{\delta}{\delta P_\mu(1)}$, so it acts on $\mathcal{C}[P]_S^1$, whereas the right one acts on $\mathcal{C}[P]_0^S$. In the same way, the left derivative $\frac{\delta}{\delta P_\mu(S-)}$ acts on $\mathcal{C}[P]_0^S$, and the right one acts on $\mathcal{C}[P]_S^1$. As a result we get all four possible terms, each with the weight $\frac{1}{4}$.

E Generating Function for Super-Graphs

There is a simple solvable model which serve as generating function for these super-graphs. Consider the sum of two *Abelian* constant gauge field strengths $G_{\mu\nu} + B_{\mu\nu}$. The superpath integral is computed in Appendix B, and it reduces to a Pfaffian

$$\int \mathcal{D}X \delta^d(X(0)) \exp \left(\frac{\imath}{2} (G_{\mu\nu} + B_{\mu\nu}) \int_0^1 dX_\nu(S) X_\mu(S) \right) \propto \sqrt{\det(B + G)}. \tag{187}$$

On the other hand, we could replace the G symplectic form by an oscillator commutation relation

$$[R_\mu, R_\nu] = \imath G_{\mu\nu}, \tag{188}$$

and compute the vacuum average instead

$$\begin{aligned}
& \int \mathcal{D}X \delta^d(X(0)) \left\langle 0 \left| \hat{T} \exp \left(\int_0^1 dX_\mu(U) R_\mu \right) \right| 0 \right\rangle_R \\
& \exp \left(\frac{\imath}{2} B_{\mu\nu} \int_0^1 dX_\nu(S) X_\mu(S) \right) \propto \sqrt{\det(B + G)}.
\end{aligned} \tag{189}$$

This formula (see [1]) follows from the definition of the the ordered product and the Baker-Hausdorff formula. Another proof is to switch to Schwinger gauge in the ordered exponential,

treating R_μ as a non-Abelian gauge field $R_\mu(S)$. This field is constant, but it has a finite field strength (188). This field strength being Abelian, the Taylor expansion in a Schwinger gauge terminates, so that one may replace

$$R_\nu(S) \Rightarrow \frac{\imath}{2} G_{\mu\nu} X_\mu(S), \quad (190)$$

which recovers the initial Abelian formula.

Comparing the terms of expansion in powers of G we get relations

$$\begin{aligned} \langle 0 | R_{\mu_1} \dots R_{\mu_{2n}} | 0 \rangle \int_0^1 \langle d^{2n} X_{\{\mu\}}(\{U\}) \rangle_{Gauss} = \\ \left(\exp \left(- \sum_{k=1}^{\infty} \frac{(-1)^k}{2k} \text{tr} (B^{-1} G)^k \right) \right)_n, \end{aligned} \quad (191)$$

with the Gauss averages

$$\langle DX_\mu(S) DX_\nu(S') \rangle = \imath B_{\mu\nu}^{-1} D\delta(S - S'). \quad (192)$$

On the right-hand side the term $O(G^n)$ is left in the Taylor expansion. The vacuum average can be computed from commutation relations, which gives the set of sum rules for the locally SUSY integrals we need.

The point is this set of relations is valid for *arbitrary* d , while the SUSY integrals are universal numbers, *independent of* d . So, these relations do not terminate at $n = 4$ in four dimensions, where we need them.

Let us be more specific. Expanding the exponential on the right side, we get

$$\sum_{l=0}^n \frac{(-1)^{n-l}}{l!} \sum_{k_1=1}^{n+1-l} \dots \sum_{k_l=1}^{n+1-l} \delta_{n, \sum k_i} \prod_{i=1}^l \frac{\text{tr} (B^{-1} G)^{k_i}}{2k_i}. \quad (193)$$

Each factor of B^{-1} corresponds to the Gauss contraction (192), and each factor of G corresponds to the commutator (188). The indexes between the B^{-1} and G tensors are contracted according to the specific Gauss diagram. The coefficient in front of this product of tensors $B^{-1} \otimes \dots B^{-1} \otimes G \otimes \dots G$ on the left is given by the SUSY integral

$$\pm \int_0^1 d^{2n} U \prod_{\langle ij \rangle} D\delta(U_i - U_j). \quad (194)$$

The same coefficient on the right is given by above sum of $\frac{(-1)^{n-l}}{l!} \prod_{i=1}^l \frac{1}{2k_i}$, independently of d . In particular, the first relation reads

$$\begin{aligned} \frac{1}{2} \text{tr} B^{-1} G &= -\imath B_{\mu\nu}^{-1} \langle 0 | R_\mu R_\nu | 0 \rangle \int_0^1 d^2 U D\delta(U_1 - U_2) = \\ &= -\frac{1}{2} \imath B_{\mu\nu}^{-1} \langle 0 | [R_\mu, R_\nu] | 0 \rangle \int_0^1 d^2 U D\delta(U_1 - U_2) = \\ &= \frac{1}{2} B_{\mu\nu}^{-1} G_{\mu\nu} \int_0^1 d^2 U D\delta(U_1 - U_2) = -\frac{1}{2} \text{tr} B^{-1} G \int_0^1 d^2 U D\delta(U_1 - U_2), \end{aligned} \quad (195)$$

from which we get

$$\int_0^1 d^2 U D \delta(U_1 - U_2) = -1. \quad (196)$$

These relations in higher orders leave some undetermined SUSY integrals.

The typical integral involves the products of the Θ functions and their derivatives, i.e. δ functions. The number of δ functions is twice less than number $2n$ of integration variables, moreover, there are δ functions for every variable, otherwise the local SUSY would be violated.

The simplest graphs are those, when each Θ function is differentiated only once. In this case all the points will collapse to one of the endpoints of the interval $0, 1$. Some Θ function will collapse as well, leading to a bubble graph at one or another endpoint,



but some other will connect the endpoints, contributing just $\Theta(0, 1) = 1$. Each bubble will give $\Theta(0, 0) = \Theta(1, 1) = \frac{1}{2}$, so that the integral would be $\pm 2^{-k}$ where k is the number of bubbles.

The generic term which arises in momentum expansion involves derivatives of the δ functions

$$\pm \int_0^1 d^n S \int_0^1 d^m U K_{\{\mu\}}(\{U\}) \prod D \delta(U_b - S_c) \quad (197)$$

where $\{1 \dots m\}$ is partitioned into $\{a\}$, $\{b\}$ and $\{c\} = \{1, \dots n\}$. There are precisely n indexes in the $\{b\}$ list, and $m - n$ in the $\{a\}$ list.

Substituting here the above definition of $K_\mu(U)$ we get superposition of terms with products of Kronecker deltas times

$$\begin{aligned} & \int_0^1 d^n S \int_0^1 d^m U \prod \delta(U_a - S_i) \prod D \delta(U_b - S_c) = \\ & \int_{-\infty}^{\infty} dS_1 \dots \int_{-\infty}^{\infty} dS_n \int_{-\infty}^{\infty} dU_1 \dots \int_{-\infty}^{\infty} dU_m \\ & \Theta(0, S_1) \dots \Theta(S_n, 1) \Theta(0, U_1) \dots \Theta(U_m, 1) \prod \delta(U_a - S_i) \prod D \delta(U_b - S_c) \end{aligned} \quad (198)$$

These superspace integrals are universal numbers, which depend on the partition of indexes. This follows from the local SUSY, which is preserved here.

The derivatives D in the last line can be integrated by parts, either over S_c or over U_b in such a way, that they act only on the Θ functions, rather than δ functions. We get then

$$\int_{-\infty}^{\infty} dS_1 \dots \int_{-\infty}^{\infty} dS_n \int_{-\infty}^{\infty} dU_1 \dots \int_{-\infty}^{\infty} dU_m \prod_l \Theta(S_l, S_{l+1}) \prod \Theta(U_k, U_{k+1}) \quad (199)$$

$$\prod \delta(U_j - U_{j+1}) \prod \delta(S_r - S_{r+1}) \prod \delta(U_a - S_i) \prod \delta(U_b - S_c)$$

The number of δ functions equals $m + n$, therefore all the integrals can be done. In case there are cycles, there will be some $\delta(0)$ factors. All such terms drop, as

$$\delta(0) = 0, \tag{200}$$

$$\delta(S - S')\delta(S' - S) = 0,$$

$$\delta(S - S')\delta(S' - S'')\delta(S'' - S) = 0, \dots$$

in superspace. So do the terms with the cycle of Θ functions, as we already discussed above. These identities can be proven by explicit calculation.

So, only terms without cycles will be left. In those terms we shall have the products like $\delta(U_i - U_{i+1})\Theta(U_i, U_{i+1})$. These products can be replaced by $\frac{1}{2}\delta(U_i - U_{i+1})$, since $\Theta(U, U) = \frac{1}{2}$.